Trade, Labor Reallocation Across Firms and Wage Inequality

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February 2024

Abstract

This paper develops a framework for studying the effects of higher trade openness on the wage distribution in which strong skill-productivity complementarities in production imply that inequality rises as workers reallocate toward more-productive (skill-intensive) firms in the same industry. The model features a large number of skill groups and weaker and more empirically relevant restrictions on firm selection into exporting than standard heterogenous-firms models. An autarkic economy that opens to trade always experiences a pervasive rise in wage inequality under no firm entry, with wage polarization being another possibility under free entry. Theoretically, more outcomes are possible following a trade liberalization in a trading economy. In a calibrated version of the framework, any increase in trade openness always leads to pervasively higher wage inequality. The analysis highlights the importance of properly accounting for the role of new exporters (extensive margin) in shaping the aggregate relative demand for skills, a channel controlled by assumptions affecting selection into exporting.

Keywords: Trade, firms, workers, supermodularity, wage inequality.
JEL codes: F10, F12, F16.

*All the views expressed in this paper are mine and do not necessarily represent those of the Federal Reserve System.

1 Introduction

Wage inequality has risen significantly in many countries since the late 70s, a period that also saw a rapid expansion of international trade. Three broad lessons follow from the empirical research exploring the connection between both trends. First, as discussed in Goldberg and Pavcnik (2007) and Helpman (2016), the rise in inequality is largely accounted for by within-industry effects, with the evidence providing little support for the between-industry channels emphasized by the traditional factor-proportions trade theory.[1] Second, firms may be an important part of the story behind the changes in the wage distribution. For example, Krishna, Poole, and Senses (2014) find substantial within-industry labor reallocation across firms following a trade liberalization that cannot be explained by a random assignment of workers to firms.[2] Third, divergent trends in inequality in different parts of the wage distribution (Autor, Katz, and Kearney 2008) and a rise in within-group (residual) wage inequality (Acemoglu 2002; Attanasio, Goldberg, and Pavcnik 2004) indicate that grouping workers into a few large skill-groups (as typically done in the literature) does not provide enough detail to understand the full distributional consequences of international trade.

In light of these lessons, this paper develops a general equilibrium trade model with a large number of skill groups that emphasizes within-industry labor reallocation across heterogeneous firms as the mechanism through which trade affects the wage distribution. In particular, strong skill-productivity complementarities in production imply that an increase in trade openness raises wage inequality when it induces a reallocation of workers toward more-productive (skill-intensive) firms in the same industry. I use the model to study the channels through which a trade-induced labor reallocation affects the wage distribution, including the entry and exit of firms into and out of the market, the increased demand of incumbent exporters, and the demand of new exporters.

The framework builds on standard heterogenous-firm trade models. As in Melitz (2003), labor is the only factor of production, the labor market is perfectly competitive, and final goods are produced by monopolistically competitive firms that differ in their productivity. In addition, the presence of fixed production and export costs leads to selection into activity and into exporting—i.e., only some firms find it optimal to produce, and only a subset of them export. Departing from Melitz (2003), the labor force comprises heterogeneous workers of a continuum of skill types, so firms must choose not only the total number of *production* workers to hire but also the mix of skill-types to employ. Strong production complementarities between worker skill and firm productivity imply that more-productive firms have workforces of higher average ability in equilibrium.

The core of the framework lies in the production and export technology of firms. The output of a firm depends linearly on the number of *production* workers of each skill type that it employs. The productivity of a production worker at a given firm is a strictly log supermodular function of the worker's skill and the firm's productivity, giving more able workers a comparative advantage in production at more-productive

[1] This evidence includes a rise in the skill-premium in developed and developing countries (Goldberg and Pavcnik 2007), and little inter-industry labor reallocation following trade liberalizations.

[2] In addition, as discussed in Card et al. (2016), numerous studies find similar trends in the aggregate dispersion of wages and firms' productivity.

firms. As in Costinot and Vogel (2010), these assumptions permit the analysis of market equilibrium to transform into the analysis of a matching problem. In particular, the equilibrium allocation of *production* workers among *active* firms is characterized by a strictly increasing and continuous matching function that maps the skill types of the former to the productivity types of the latter. Moreover, this matching function is a sufficient statistic for the dispersion wages in this setting, facilitating the analysis of comparative static predictions about wage inequality.

Fixed export costs also play an important role, as they determine firm selection into exporting, shaping the set exporters and their collective demand for skills. Therefore, I consider a flexible specification of fixed export costs that can accommodate weaker and more empirically relevant restrictions on firm selection into exporting than standard heterogeneous-firms trade models.[3] Specifically, I posit that fixed export costs vary across firms, and model their firm-specific sizes as independent realizations of a nonnegative random variable with an absolutely continuous and increasing cumulative distribution fuction (CDF). As a result, exporters are, on average, more productive than nonexporters in equilibrium, but high-productivity nonexporters coexists with low-productivity exporters. Finally, all fixed costs are paid in terms of a "skill bundle" that comprises *nonproduction* workers of all skill levels, an assumption that allows me to isolate the impact on the wage distribution of the endogenous assignment of production workers to firms.

The cross section of the model captures several features of the data identified by the trade and labor literatures. The dispersion of wages in the model reflects between-firms wage differences (rather than within-firm differences), a channel that represents around 60% of the wage dispersion in the United States (Davis and Haltiwanger 1991). In addition, more-productive firms tend to be larger (in terms of output), have workforces of higher average ability, and pay higher average wages (Card et al. 2016). Per the stochastic representation of fixed export costs, the model features an imperfect positive correlation between size, firm wages and export status (Bernard and Jensen 1995) as well as between the latter and firm productivity, leading to overlapping productivity distributions for exporters and nonexports (Bernard, Eaton, Jensen, and Kortum 2003). Finally, if workers are classified in large skill groups, possibly reflecting imperfect observability of worker ability, then the model features wage heterogeneity within each of these skill groups (Acemoglu 2002; Attanasio et al. 2004).

I carry out the analysis of the effects of trade on the wage distribution under two widely used assumptions about firm entry into the industry: no free entry a-lá Chaney (2008) and free entry a-lá Melitz (2003). These alternative entry assumptions lead to the no-free-entry and free-entry models analyzed in the paper, whose predictions can be interpreted, respectively, as the short- and long-term effects of trade.[4] These models differ only in the equilibrium condition that pins down the *activity cutoff*, the productivity value below which firms do not find it profitable to produce. Conditional on the activity cutoff, the two models are identical, so they share the cross-sectional features discussed above.

To study the impact of higher trade openness on the wage distribution, I decompose the associated

[3]Assuming common fixed export costs across firms has been standard since Melitz (2003). This unrealistic assumption is not inocuous in this setting as it affects the distributional effects of trade.

[4]Exploring the implications of these two alternative entry assumptions also serves a pedagogical purpose. By delivering sharper results, the no-free-entry model facilitates the analysis of the main forces at play, which in turn simplifies the discussion of the more nuanced implications of the free-entry model.

labor reallocation across firms into three channels. The first channel, the *selection-into-activity* channel, captures the reallocation of resources driven by changes in the set of active firms—i.e., by changes in the activity cutoff. The second channel, the *intensive margin* of trade, reflects the changes in the production and employment decisions of incumbent exporters that continue serving the foreign market after the decline in trade frictions. Finally, the third channel, the *extensive margin* of trade, captures the reallocation of employment associated with changes in the set of exporters. These last two channels are largely determined by firm selection into exporting, highlighting the importance of not arbitrarily restricting this margin of adjustment in the model. This decomposition not only highlights the key elements driving the results in the current setting, but also facilitates the comparison with the implications of other frameworks in the literature exploring the connection between international trade, firms, and wages.

I analyze two instances of increased trade openness, opening to international trade and a trade liberalization, where the latter is defined as a decline in the variable trade costs faced by an economy that already participates in international trade. In the *no-free-entry model*, an initially autarkic economy that opens to trade always experiences an increase in the activity cutoff and a pervasive rise in wage inequality, in the sense that for any pair of workers, the relative wage of the more-skilled one rises. In terms of the three channels discussed above, the selection-into-activity channel induces a pervasive rise in wage inequality, as the exit of the least productive (low-skill-intensive) firms leads to a decline in the relative demand of less-skilled workers. With no exporters in the initial autarkic equilibrium, the intensive margin channel is not operational in this counterfactual. Finally, the extensive margin channel also leads to a pervasive rise in wage inequality; the (new) exporters in the open economy are, on average, more productive than nonexporters, so their collective labor demand is biased toward more-skilled workers. The importance of this channel, which depends on how fast the fraction of exporting firms increases with productivity, is determined by the CDF of fixed export costs.

A trade liberalization can lead to additional outcomes. Although a decline in variable trade costs in the no-free-entry model always leads to an increase in the activity cutoff and a rise in wage inequality at the lower end of the wage distribution, little can be said about its impact elsewhere in the distribution. As the activity cutoff rises, the selection-into-activity channel leads to a pervasive rise in wage inequality. The intensive-margin channel also leads to a pervasive rise in wage inequality, reflecting a rise in the more-skill-intensive labor demand of incumbent exporters as they expand their production to satisfy a higher foreign demand. In contrast, the impact of the extensive-margin channel on the wage distribution is theoretically ambiguous. Without additional restrictions on the CDF of fixed export costs, new exporters can be (on average) more or less productive than incumbent firms, so their collective demand may be biased toward more- or less-skilled workers. Moreover, the ambiguity about the effects of this third channel extends to the overall impact of a trade liberalization on the wage distribution. This result highlights the importance of paying close attention to the modeling of the extensive-margin channel in any study emphasizing the role of heterogenous firms in the distributional consequences of higher trade openness. I present sufficient conditions on the CDF of export costs under which wage inequality rises pervasively after a trade liberalization.

Assuming *free entry* brings an additional source of ambiguity relative to the previous results, as the effects of increased trade openness on the activity cutoff cannot be determined without imposing additional restrictions on primitives. If the *activity cutoff rises* after the economy opens to trade or after a liberalization, then the distributional effects predicted by the free-entry model are qualitatively the same as those described earlier for the no-free-entry model. If the *activity cutoff declines* after the economy opens to trade, then wages polarize—i.e., wage inequality decreases among the least-skilled workers but increases among the most-skilled ones. In this case, the selection-into-activity and extensive-margin channels lead to a pervasive decline and a pervasive rise in wage inequality, respectively, with the former channel dominating at the lower end of the wage distribution and the latter at the upper end. Finally, if a trade liberalization leads to a decline in the activity cutoff, then wage inequality necessarily decreases at the lower end of the distribution and increases somewhere else, but additional outcomes beyond wage polarization are possible. Of note, regardless of entry assumptions, an increase in trade openness never leads to a pervasive decline in wage inequality in this framework.

I also explore the effects of higher trade openness on the level of real wages. For both entry assumptions, an increase in trade openness (opening to trade or liberalization) always raises average real wages, but the least-skilled workers in the economy could see their real wage decline. In the free-entry-model, the fate of the real wages of these workers is completely determined by the response of activity cuttoff, leading to interesting connections between the effects of higher trade openness on the level and distribution of wages. For example, opening to international trade raises the real wage of the poorest workers in the economy only if it also induces a pervasive rise in wage inequality.

To assess the empirical relevance of the theoretical possibilities described above, I calibrate the model based on estimates from the literature and some broad features of firm data from Portugal. Given its informational content about the extensive-margin channel in the model, which drives much of the ambiguity in the theoretical results, a crucial target of the calibration is the fraction of firms that export in each decile of the empirical distribution of firms by value added per worker. For both entry assumptions, the calibrated model predicts pervasively higher wage inequality and higher real wages for all workers following any increase in trade openness. In the case of a trade liberalization, wage inequality always increases through the selection-into-activity and intensive-margin channels, while it decreases slightly through the extensive-margin channel. These results suggest that a decline in trade costs is likely to lead to pervasively higher wage inequality, in both the short and long run, through the labor-reallocation mechanisms emphasized in this paper. The analysis also highlights the importance of accurately quantifying the extensive-margin channel in the model. Indeed, assuming common fixed export costs across firms, as has been standard since Melitz (2003), results in much larger distributional effects through this channel, leading in some cases to declines in inequality in some parts of the wage distribution following a liberalization.

This paper is related to a growing number of studies using assignment models to study the distributional consequences of international trade and offshoring. Studies based on two-region competitive models, such as Grossman and Maggi (2000), Ohnsorge and Trefler (2007), Antràs, Garicano, and Rossi-Hansberg (2006), and Costinot and Vogel (2010), emphasize differences in higher moments of the skill distribution across regions as the drivers of trade and its distributional effects, with these effects generally differing qualitatively across regions as a result.[5] In contrast, different countries can experience similar distributional effects from trade through the mechanisms emphasized in this paper, as they do not rely on differences across countries. As such, the framework in this paper is better suited to think about the expansion international trade as a common factor contributing to the rise in wage inequality observed in many economies since the late 70s.

Methodologically, this paper is closer to a branch of this literature that, building on Costinot (2009), develops two-sided heterogeneity models by embedding in different general equilibrium frameworks a production technology similar to the one considered in this paper, giving rise to similar assignment problems. In models with neoclassical roots, Costinot and Vogel (2010) study the assignment of workers to tasks while Grossman, Helpman, and Kircher (2017) study the matching of managers and workers and their sorting into different industries.[6] In monopolistically competitive settings, Sampson (2014) and Somale (2015) analyze the assignment of workers to firms in models that extend Yeaple (2005) and Chaney (2008), respectively. However, a general equilibrium analysis of a similar extension of Melitz (2003), the canonical heterogeneous-firm trade model, has proved technically challenging.[7] I contribute to this literature by presenting said analysis under weaker assumptions about selection into exporting and by showing how this type of models can be taken to the data.

This paper contributes methodologically to this branch of the assignment trade literature by deriving a set of lemmas and propositions that facilitate the general equilibrium analysis of models featuring similar assignment problems. Among other results, I establish the existence and uniqueness of the equilibrium, a prerequisite for a theoretical analysis of comparative statics. Conditional on the activity cutoff, the market equilibrium is characterized by a system of nonlinear differential equations and a set of boundary conditions that together define a nonlinear two-point boundary value problem (BVP). In contrast to the cases of initial value problems (IVP) and linear BVPs, establishing existence and uniqueness of solutions is not trivial in the case of nonlinear BVPs, with off-the-shelf mathematical results typically covering particular cases of the problem. Despite these difficulties, several studies in the trade literature that use assignment models leading to similar BVPs simply assume or state without proof the existence and uniqueness of the solution. In this paper, I fill this gap in the trade literature for the case of a nonlinear two-point boundary BVP that encompasses those in this paper and others in the literature.[8]

This paper also relates to a literature proposing heterogeneous-firms models in which international trade can affect wage inequality through within-industry mechanisms. Motivated by developments in within-group wage inequality, one line of research develops models with labor market frictions in which ex-ante identical workers earn different wages at different firms, reflecting differences in efficiency wages (Davis and Harrigan 2011) or fair wages (Egger and Kreickemeier 2009, 2012; Amiti and Davis 2012) required to induce worker effort, as well as differences in average ex-post worker ability amid search-and-

[5] Of note, trade among identical countries has no distributional effects.

[6] The distributional effects of trade in these tudies also relies on differences across countries.

[7] Aducing intractability, Sampson (2014) presents only some partial equilibrium results in this setting. In Somale (2015), I only considered the effects of opening to trade under no free entry and common fixed export costs accross firms.

[8] The general BVP considered in this paper encompasses those in Costinot and Vogel (2010), Sampson (2014), Somale (2015), Grossman, Helpman, and Kircher (2017).

matching frictions, unobservable worker ability and costly screening (Helpman, Itskhoki, and Redding 2010; Helpman et al. 2016). Given their focus on ex-ante indentical workers, these models cannot speak to the effects of trade on the relative reward to observable worker characteristics, such as the effects on the skill premium. By contrast, the framework in this paper can speak to these issues as well as to within-group inequality if workers are classified in large skill groups.

Another strand of this literature focuses on the effects of trade on the relative earnings of ex-ante heterogeneous workers (from the perspective of firms) through firms' technological choices (Yeaple 2005; Bustos 2011; Sampson 2014), workers' occupational choices (Monte 2011) or changes in the distribution of labor demand across firms differing in skill intensity (Somale 2015, Burstein and Vogel 2017). While these studies typically contemplate only a few large skill groups or place strong restrictions on selection into exporting, I consider a continuum of skill groups and a flexible specification of the latter.[9] This allows me to study the effects of trade on the entire wage distribution under empirically relevant restrictions on selection into exporting, showing that restrictions typically imposed on this margin can lead to significantly different distributional effects.

The rest of the paper is organized as follows. Section 2 describes the basic setup of the framework. Sections 3 and 4 characterize the equilibrium in the no-free-entry model and present existence and uniqueness results. Section 5 studies the effects of higher trade openness on wage inequality in the no-free-entry model, while section 6 extends the analysis to the free-entry model. After describing the calibration approach, section 7 discusses the implications of a calibrated version of the model. Section 8 concludes.

2 Basic Setup

This section develops a framework for studying the effects of higher trade openness on the wage distribution in which strong skill-productivity complementarities in production imply that inequality rises as workers reallocate towards more productive firms in the same industry. The model features a large number of skill groups and a flexible specification of fixed export costs that can accommodate weaker and more empirically relevant restrictions on firm selection into exporting than standard heterogeneous-firms trade models.

2.1 Demand

The preferences of the representative consumer are given by a C.E.S utility function over a continuum of goods indexed by ω :

$$U = \left[\int_{\omega\in\Omega} u(\omega)^{\frac{\sigma-1}{\sigma}}\, d\omega\right]^{\frac{\sigma}{\sigma-1}},$$

where $u(\omega)$ is the quantity consumed of good ω, the measure of the set Ω represents the mass of available goods and $\sigma > 1$ is the elasticity of substitution between goods. The demand and expenditure for

[9] These strong restrictions are introduced by assuming common fixed export costs across firms in models based on Melitz (2003) and by imposing strong functional form assumptions on productivity distributions in models based on Eaton and Kortum (2002).

individual varieties generated by this utility function are

$$u(\omega) = EP^{\sigma-1}p(\omega)^{-\sigma}, \qquad E(\omega) = EP^{\sigma-1}p(\omega)^{1-\sigma}, \tag{1}$$

where P is the aggregate price level and E is aggregate expenditure,

$$P = \left[\int_{\omega\in\Omega} p(\omega)^{1-\sigma}\, d\omega\right]^{\frac{1}{1-\sigma}}, \qquad E = \int_{\omega\in\Omega} E(\omega)\, d\omega. \tag{2}$$

2.2 Production

There is a continuum of *active*, monopolistically competitive firms in the market, each producing a different variety ω.[10] As in Melitz (2003), firms differ in their productivity level ϕ, which they obtain as an independent draw from a distribution $G(\phi)$ with density function $g(\phi)$. I assume that the support of G, $\Phi \equiv \{\phi : g(\phi) > 0\}$, is equal to some bounded interval of nonnegative real numbers, $\left[\underline{\phi}, \overline{\phi}\right] \subseteq \mathbb{R}_+$. In contrast to Melitz (2003), the labor force is heterogenous, consisting of a continuum of workers of mass L that differ in their skill level s. The distribution of worker's skills is represented by a nonnegative density $V(s)$, so $LV(s) \geq 0$ represents the inelastic supply of workers with skill s. I only consider skill distributions such that the support of V, denoted by S, is equal to some bounded interval of nonnegative real numbers—i.e., $S \equiv \{s : V(s) > 0\} = [\underline{s}, \overline{s}] \subseteq \mathbb{R}_+$.

The production technology of firms is represented by a cost function that exhibits constant marginal cost and fixed overhead costs. After paying the fixed costs described below, a firm must decide the mix of workers to use in production. The total output of a firm with productivity ϕ, $q(\phi)$, is given by

$$q(\phi) = \int_{s\in S} A(s,\phi)\, l(s,\phi)\, ds, \tag{3}$$

where $A(s,\phi)$ is the marginal productivity of a worker of skill s, and $l(s,\phi)$ is the total number of *production* workers of that skill level employed by the firm.[11] More skilled workers are more productive than less skilled workers, regardless of the productivity of the firm that employs them. Also, more productive firms have lower labor input requirements than less productive firms no matter the type of worker considered. In terms of the production function (3), I formally assume that the productivity function $A(s,\phi)$ is strictly positive, strictly increasing and continuously differentiable—i.e., $A(s,\phi) > 0$, $A_s(s,\phi) > 0$ and$A_\phi(s,\phi) > 0$.[12]

In addition to the absolute productivity advantage described above, more skilled workers have a comparative advantage in production at more productive firms. Specifically, I follow Costinot and Vogel (2010) and assume that the function $A(.,.)$ is strictly log-supermodular, $A(s',\phi')\, A(s,\phi) > A(s',\phi)\, A(s,\phi')$ for all $s' > s$ and $\phi' > \phi$.Since $A(s,\phi) > 0$, the previous inequality can be rearranged as $\frac{A(s',\phi')}{A(s',\phi)} > \frac{A(s,\phi')}{A(s,\phi)}$, showing that the productivity gains from switching to a more productive firm are higher for more skilled

[10] A firm is active in the market if it produces positive output.

[11] Firms also employ nonproduction workers as part of their fixed costs requirements.

[12] For any function $F(x_1, ..., x_n)$, F_{x_i} denotes the partial derivative of F with respect to variable x_i.

workers. Alternatively, the gains from hiring a more skilled worker are higher for more productive firms.

Following a standard practice in the international trade literature, I assume that fixed costs are paid in terms of labor. Specifically, I assume that firms pay a fixed cost of $fV(s)$ units of each skill $s \in S$, implying that the total fixed cost of a firm is $f \int_{\underline{s}}^{\overline{s}} w(s) V(s) ds = f\overline{w}$, where $w(s)$ is the wage of a worker with skill level s, and $\overline{w}$ is the average wage in the economy and the numeraire, $\overline{w} = 1$. This specification of fixed costs guarantees that the distribution of skills in the economy is still given by $V(s)$ after all fixed costs have been paid, implying that the demand of labor induced by fixed-costs requirements has no effect on the wage schedule $\{w(s)\}$. As such, the wage schedule is completely determined by the interactions between the exogenous relative supply of skills, captured by the distribution $V(s)$, and the endogenous relative demand of skills derived from the firm's demand of *production* workers.

2.3 Variable Costs and Prices

Per the linear production technology (3), workers are perfect substitutes in production. Accordingly, firms employ only those worker types that entail the lowest cost per unit of output, implying that the marginal cost of a firm with productivity ϕ, $c(\phi)$, is given by

$$c(\phi) = \min_{s \in S} \left\{ \frac{w(s)}{A(s, \phi)} \right\}. \tag{4}$$

For any wage schedule, the marginal cost $c(\phi)$ is strictly decreasing in the productivity level ϕ, as a firm can always hire the same type of workers employed by a less-productive competitor and obtain a strictly lower marginal cost due its absolute productivity advantage, $\phi' > \phi \Leftrightarrow c(\phi') < c(\phi)$.

Faced with the iso-elastic demands in (1), firms optimally set their price equal to a constant markup over their marginal costs, $p(\phi) = \frac{\sigma}{\sigma - 1} c(\phi)$. This pricing rule and the cost minimization condition (4) imply

$$p(\phi) \leq \frac{\sigma}{\sigma - 1} \frac{w(s)}{A(s, \phi)} \text{ for all } s \in S; \qquad p(\phi) = \frac{\sigma}{\sigma - 1} \frac{w(s)}{A(s, \phi)} \text{ if } l(s, \phi) > 0. \tag{5}$$

2.4 Entry

I carry out the analysis under two widely-used assumptions regarding entry; no free entry a-lá Chaney (2008) and free entry a-lá Melitz (2003). In the first case, there is a fixed mass of firms in the industry. In the second case, there is unbounded pool of prospective firms that must pay a fixed entry cost to develop a new product variety and enter the industry. The results obtained under the no-free-entry assumption can be interpreted as the short-term consequences of trade, before investment in the development of new varieties leads new firms to enter the industry. In contrast, the results obtained under the free-entry assumption can be viewed as the long-term effects of trade.

3 No-Free-Entry Model: the Closed Economy

As in Chaney (2008), there is a fixed mass $\overline{M}$ of firms in the industry. A firm is *active* in the market if and only if it finds it profitable to produce. The pricing rule (5), the consumer's demand and expenditure functions in (1), and the goods-market clearing condition ($u(\omega) = q(\omega)$), imply that a firm's output, revenue and profit from serving the *domestic* market are given by

$$q^d(\phi) = EP^{\sigma-1}\left[\frac{\sigma}{\sigma-1}c(\phi)\right]^{-\sigma};\ r^d(\phi) = EP^{\sigma-1}\left[\frac{\sigma}{\sigma-1}c(\phi)\right]^{1-\sigma};\ \pi^d(\phi) = \frac{r^d(\phi)}{\sigma} - f, \tag{6}$$

where aggregate expenditure, E, equals aggregate income. The last expression, together with a decreasing marginal cost function $c(\phi)$, implies that a firm's profit is an increasing function of the firm's productivity.

There are combinations of parameters such that all firms are active in equilibrium, $\pi^d(\underline{\phi}) \geq 0$. However, since this case is not theoretically interesting nor empirically relevant, I focus on equilibria featuring selection into activity—i.e., the least-productive firms find it unprofitable to produce and remain inactive, $\pi^d(\underline{\phi}) < 0$.[13] In such an equilibrium, there is a cutoff productivity value $\phi^* \in (\underline{\phi}, \overline{\phi})$ such that only firms with productivity above this value are active in the market. The value of this *activity cutoff* corresponds to the level of productivity at which firms make zero profits,[14]

$$\pi^d(\phi^*) = 0. \tag{7}$$

In turn, the activity cutoff ϕ^* determines the total mass of active firms in the industry,

$$M = [1 - G(\phi^*)]\,\overline{M}. \tag{8}$$

Finally, the labor market of each type of worker must clear,

$$LV(s) = \int_{\phi^*}^{\overline{\phi}} l^d(s,\phi)\frac{g(\phi)}{[1-G(\phi^*)]}d\phi M + MfV(s) \text{ for all } s \in S. \tag{9}$$

The left- and right-hand sides of the last expression capture, respectively, the total supply and demand of workers of skill s, with the total demand comprising the demand of *production* workers (first term), and the demand of *nonproduction* workers derived form the presence of fixed costs of production (second term). Having described all the components of the economy, I state the formal definition of the equilibrium.

Definition 1 *A no-free-entry equilibrium of the closed economy is a mass of active firms $M > 0$, a productivity activity-cutoff , $\phi^* \in (\underline{\phi}, \overline{\phi})$, an output function $q^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a labor allocation function $l^d : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a price function $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and a wage schedule $w : S \to \mathbb{R}_+$ such that the following conditions hold,*[15]
(i) consumers behave optimally, equations (1) and (2);

[13] Proposition 1 presents conditions on primitives that rule out this possibility.
[14] If all firms are active in the market, then $\phi^* = \underline{\phi}$, and condition (7) may not hold.
[15] Technically, this definition corresponds to an equilibrium featuring selection into activity.

(ii) firms behave optimally given their technology, equations (3), (5), (7) and (8);

(iii) goods and labor markets clear, equations (6) and (9), respectively;

(iv) the numeraire assumption holds, $\overline{w} = 1$.

3.1 Characterization of the Equilibrium

The log-supermodularity of the productivity function, A, implies that the equilibrium labor allocation is characterized by positive assortative matching—i.e., more-productive firms employ *production* workers of higher ability. Specifically, there exists a continuous and strictly increasing matching function $N : S \rightarrow [\phi^*, \overline{\phi}]$ such that, all firms of productivity $N(s)$ employ production workers of skill s, and all production workers of skill s are employed at firms with the productivity $N(s)$. Behind this result, formally stated in lemma 1, lies a simple intuition. The cost-minimization condition (4) implies that a firm of productivity ϕ' employing a worker of skill s' cannot reduce its marginal cost of production by employing a worker of a different skill, that is, $w(s')/A(s', \phi') \leq w(s)/A(s, \phi')$ for all $s \in S$. This observation and the strict log-supermodularity of A imply that, for any skill level $s > s'$ and any productivity level $\phi < \phi'$, the following inequalities hold, $\frac{A(s,\phi)}{A(s',\phi)} < \frac{A(s,\phi')}{A(s',\phi')} \leq \frac{w(s)}{w(s')}$. Accordingly, a firm with productivity $\phi < \phi'$ does not employ workers of skill $s > s'$, as it can obtain a strictly lower marginal cost by hiring a worker of skill s'. Although this argument only proves that the matching function is weakly increasing, it highlights the connection between the log-supermodularity of A and positive assortative matching in equilibrium.

Armed with the previous result, the equilibrium can be characterized in terms of the matching function N, revealing a tight connection between the latter and wage inequality in the current framework. A worker of skill s is matched to a firm with productivity $N(s)$ in equilibrium if and only if the skill level s solves the cost minimization problem (4) for any firm with productivity $\phi = N(s)$. The first order condition for an interior solution of this problem yields the following equilibrium condition,[16]

$$\frac{d \ln w(s)}{ds} = \frac{\partial \ln A(s, N(s))}{\partial s}. \tag{10}$$

The last expression is central in the analysis of wage inequality. It implies that the matching function N is a sufficient statistic for the dispersion of wages in the economy, as it is the only endogenous variable affecting the slope of the wage schedule. The connection between N and wage inequality can be seen more clearly by integrating (10) between s' and $s'' > s'$ to get $w(s'')/w(s') = \exp\{\int_{s'}^{s''} \frac{\partial \ln A(t,N(t))}{\partial s} dt\}$. The last expression, together with the strict log-supermodularity of A, implies that the ratio $w(s'')/w(s')$ is increasing in the values that the matching function takes on the interval $[s', s'']$. Then, any change in the environment leading to an upward shift of the matching function on a given interval also leads to higher relative wages for more-skilled workers in that interval. Moreover, the new distribution of wages in the interval is second-order stochastically dominated by the old one, so inequality is pervasively higher after the change.[17]

Letting $H : [\phi^*, \overline{\phi}] \rightarrow S$ denote the inverse function of the matching function N, the optimal pricing

[16] As stated in lemma 1, all the endgogenous functions considered in this section are differentiable.

[17] In appendix B.1.2, I show that the new distribution is Lorenz dominated by the previous one.

rule (5) and the expression for revenues in (6) can be used to express firm's prices and revenues as functions of the productivity level ϕ and the value of the function H at that productivity level. Totally differentiating these functions with respect to ϕ and using equation (10) in the resulting expressions yields

$$p_\phi(\phi) = -p(\phi)\frac{\partial \ln A(H(\phi),\phi)}{\partial \phi}, \tag{11}$$

$$r^d_\phi(\phi) = (\sigma-1)\, r^d(\phi)\frac{\partial \ln A(H(\phi),\phi)}{\partial \phi}. \tag{12}$$

The last two equations imply that the equilibrium matching of workers and firms is also a sufficient statistic for the dispersion of firms' prices and revenues. In particular, integrating equation (12) reveals that for $\phi'' > \phi'$, the ratio of revenues $r^d(\phi'')/r^d(\phi')$ is increasing in the values that the *inverse* of the matching function takes on $[\phi', \phi'']$, so a shift in the matching function will have opposite effects on the dispersion of wages and revenues in the closed economy.

The equilibrium labor allocation must be consistent with market clearing in the labor and goods markets—i.e., N (or H) must be consistent with conditions (1), (3), (6) and (9). This consistency requirement yields the following equilibrium condition,

$$H_\phi(\phi) = \frac{r^d(\phi)\, g(\phi)\, \overline{M}}{A(H(\phi),\phi)\left[L - f\left[1-G(\phi^*)\right]\overline{M}\right] V(H(\phi))\, p(\phi)}, \tag{13}$$

which, after some re-arrangement, states that consumers' expenditure accruing to firms with productivity ϕ, $r^d(\phi)\, g(\phi)\, \overline{M}$, must equal the total value of the output that those firms can produce with the workers they employ.

Given the equilibrium activity cutoff, ϕ^*, equations (11)-(13) form a system of nonlinear differential equations that the price function, p, the revenue function, r^d, and the inverse of the matching function, H, must satisfy in equilibrium. As is well-known, there is an uncountable family of functions that satisfy a system like (11)-(13), so a set of *boundary conditions* is needed to pin down a particular solution. Two of these boundary conditions are provided by the labor market clearing condition, as all workers must be assigned to some firm in equilibrium, $H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$. A third boundary condition is provided by the zero-profit condition for firms with productivity ϕ^*, $r^d(\phi^*) = \sigma f$. Finally, the activity cutoff ϕ^* can be determined from the the following equilibrium condition,

$$\tfrac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} r^d(\phi) g(\phi)\, d\phi \overline{M} + f\left[1-G(\phi^*)\right]\overline{M} = L, \tag{14}$$

which states that the total wages paid by firms to production and nonproduction workers (left) equals total labor income in the economy, where the expression for the latter uses the numeraire assumption. I summarize the results in this section in the following lemma.

Lemma 1 *In a no-free-entry equilibrium of the closed economy there exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$ (with inverse function H) such that (a) $l^d(s,\phi) > 0$ if and* *only if $N(s)=\phi$, (b) $N(\underline{s})=\phi^*$, and $N(\overline{s})=\overline{\phi}$. In addition, the following conditions hold*

(i) The wage schedule w is continuously differentiable and satisfies (10).

(ii) The price, revenue and matching functions, $\{p, r^d, N(\text{and } H)\}$, are continuously differentiable. Given ϕ^*, the triplet $\{p, r^d, H\}$ solves the boundary value problem (BVP) comprising the system of differential equations (11)-(13) and the boundary conditions $r^d(\phi^*)=\sigma f$, $H(\phi^*)=\underline{s}$, $H(\overline{\phi})=\overline{s}$.

(iii) The activity cutoff ϕ^* and the revenue function r^d satisfy (14).

Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$, and functions $p, r^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $H : [\phi^*, \overline{\phi}] \to S$ satisfy conditions (ii)-(iii), then they are, respectively, the productivity activity-cutoff, the price function, the revenue function, and the inverse of the matching function of a no-free-entry equilibrium of the closed economy.

4 No-Free-Entry Model: the Open Economy

Balanced trade takes place between $n+1$ symmetric (identical) economies of the type described above, so the description presented in section 2, including equations (1)-(5), holds for each of these economies. Given that the symmetry assumption ensures that all countries share the same equilibrium variables, I restrict the analysis to the home country. Firms face fixed and variable trade costs. Per-unit trade costs are common to all firms and are modeled in the standard iceberg formulation, whereby $\tau > 1$ units of a good must be shipped in order for 1 unit to arrive in a foreign destination. In contrast, fixed export costs vary across firms. A firm that wishes to export to country i must incur an idiosyncratic fixed cost of y units of a "bundle of skills" comprising $f^x V(s)$ workers of each skill $s \in S$. With the average wage as the numeraire, the total fixed export cost of the firm is $f^x y$ per foreign market. I model the firm-specific *size* of fixed export costs, y, as the realization of a nonnegative random variable Y with CDF F, which I assume is independent of the productivity distribution, absolutely continuous, and satisfies $F(y)=0$ for $y \le \underline{y}$, $dF(y) > 0$ for $y \ge \underline{y}$, where $\underline{y}$ is the lower bound of the support of Y. In addition, I assume that $f_x \underline{y} \tau^{\sigma-1} > f$, which guarantees that a firm's profit in the domestic market is always higher than in any individual foreign market.[18]

These assumptions about fixed export costs have three important implications. First, as in the case of fixed production costs, formulating fixed export costs in terms of said bundle of skills guarantees that the demand of labor induced by fixed-export-costs requirements does not affect the wage schedule. Second, in the presence of heterogeneous fixed export costs, a highly productive firm may not find it profitable to export if it faces high fixed export costs, while a less productive competitor may choose to serve the foreign market if its fixed export costs are sufficiently low. As a result, the productivity distributions of exporters and nonexporters overlap in equilibrium, consistent with the evidence in Bernard, Eaton, Jensen, and Kortum (2003). Third, an implication of the restriction $f_x \underline{y} \tau^{\sigma-1} > f$ is that, as in Melitz (2003), the activity status of a firm in the open economy continues to be determined by its domestic profit. Although not essential for the qualitative results in the paper, this implication simplifies the exposition.[19]

[18] A similar relationship between domestic and foreing profits is featured in Melitz (2003).

[19] Alternatively, I could have just assumed that a firm is active if and only if it makes positive profits in the domestic

The determination of the set of active firms and their operations in the domestic market are little changed relative to the closed economy. There is a fixed mass $\overline{M}$ of *potential* firms in the industry. A firm is *active* if and only if it makes nonnegative profits in the domestic market. The pricing rule (5) and the expenditure functions in (1) imply that the *potential* domestic output, q^d, revenue, r^d, and profit, π^d, of a firm with productivity ϕ are still given by (6). As before, domestic profits are strictly increasing in ϕ, so the equilibrium is characterized by a cutoff productivity level, $\phi^* \in (\underline{\phi}, \overline{\phi})$, such that a firm is active in the market if and only if its productivity is above this level.[20] Firms with productivity ϕ^* make zero domestic profit, condition (7), while the mass of active firms, M, is given by (8).

The equilibrium in the open economy features selection into trade—i.e., only a subset of active firms export. An active firm serves a foreign market if and only if it can make nonnegative profits there. In the presence of variable trade costs, consumers in each country face higher prices for imported goods, $p^x(\phi) = \tau p(\phi)$, so conditions (5) and (1) and the symmetry assumption imply that the *potential* export output, revenue and profit of a firm with productivity ϕ and fixed export costs $f^x y$ are given by

$$q^x(\phi) = \tau^{1-\sigma} q^d(\phi), \; r^x(\phi) = \tau^{1-\sigma} r^d(\phi), \; \pi^x(\phi) = \frac{\tau^{1-\sigma} r^d(\phi)}{\sigma} - f^x y. \tag{15}$$

Then, such a firm exports if and only if $y \leq \tau^{1-\sigma} r_d(\phi)/\sigma f^x$, which, together with the assumptions about y, implies that only a fraction $F\left(\tau^{1-\sigma} r_d(\phi)/\sigma f_x\right)$ of firms with productivity $\phi \geq \phi^*$ export. Note that this fraction is a continuous and increasing function of the productivity level ϕ, so exporters are, on average, more productive than nonexporters. These observations imply that the mass of exporters with productivity ϕ is

$$M^x(\phi) = g(\phi) F\left(\tau^{1-\sigma} r^d(\phi)/\sigma f^x\right) \overline{M}. \tag{16}$$

Finally, the labor market of each type of worker must clear,

$$\begin{aligned} LV(s) = \textstyle\int_{\phi^*}^{\overline{\phi}} [l^d(s,\phi) g(\phi) \overline{M} + l^x(s,\phi) M^x(\phi)] d\phi + \cdots \\ \cdots f M V(s) + \textstyle\int_{\phi^*}^{\overline{\phi}} n f^x \int_0^{\frac{\tau^{1-\sigma} r^d(\phi)}{\sigma f_x}} y dF(y) g(\phi) \overline{M} d\phi V(s). \end{aligned} \tag{17}$$

The left- and right-hand sides of the last expression capture, respectively, the total supply and demand for workers of skill s. Total demand comprises the demand of *production* workers to supply the domestic and foreign markets, first term, and the demand of *nonproduction* workers derived form the presence of fixed costs of production and fixed export costs, the second and third terms. Conditions (1)-(3), (5)-(8), (15)-(17) and the numeraire assumption completely describe the equilibrium, prompting the formal definition of equilibrium in the appendix, analogous to that for the closed economy.

market, regardless of its potential export profits.

[20] As before, I focus on equilibria featuring selection into activity, i.e. $\pi^d(\underline{\phi}) < 0$.

4.1 Characterization of the Equilibrium

The equilibrium of the open economy shares several features with its closed-economy counterpart. Cost minimization by firms and the strict log-supermodularity of A imply that the equilibrium labor allocation in the open economy is characterized by a strictly increasing matching function, N, that maps the set of skills, S, to the set of productivity levels of active firms, $[\phi^*, \overline{\phi}]$. In addition, equation (10), connecting the wage schedule to the matching function, and equations (11) and (12), connecting the price and domestic-revenue functions to the *inverse* of the matching function, H, continue to hold. As before, these equilibrium conditions imply that the matching function N (and its inverse H) is a sufficient statistic for the dispersion of wages, prices and domestic revenues.

The equilibrium labor allocation must be consistent with labor and goods markets clearing—i.e., N (or H) must be consistent with conditions (3), (6), (15) and (17). This observation and the expression for the mass of exporters, equation (16), yield the following equilibrium condition,

$$H_{\phi}\left(\phi\right) = \frac{r^d(\phi)\left[1+F\left(\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}\right)n\tau^{1-\sigma}\right]g(\phi)\overline{M}}{A(H(\phi),\phi)V(H(\phi))p(\phi)\left[L-fM-\int_{\phi^*}^{\overline{\phi}} nf^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF(y)g(\phi')\overline{M}d\phi'\right]}. \quad (18)$$

After some re-arrangement, the last expression states that the total revenue that firms with productivity ϕ make from their sales in the domestic and foreign markets, the numerator on the right-hand side of (18), must equal the total value of the output that those firms can produce with the workers they employ.

Given the equilibrium activity cutoff, ϕ^*, equations (11), (12) and (18) form a system of nonlinear differential equations that the price function, p, the domestic revenue function, r^d, and the inverse of the matching function, H, must satisfy in equilibrium. Two boundary conditions for this system are provided by the labor market clearing condition, as all workers must be assigned to some firm in equilibrium, $H\left(\phi^*\right) = \underline{s}$, $H\left(\overline{\phi}\right) = \overline{s}$. A third boundary condition is provided by the zero-domestic-profit condition for firms with productivity ϕ^*, $r^d\left(\phi^*\right) = \sigma f$. Finally, the open-economy counterpart of equation (14) can be used to determine the activity cutoff ϕ^*,

$$\begin{array}{c} \frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} r^d(\phi)[1+F\left(\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}\right)n\tau^{1-\sigma}]g\left(\phi\right)d\phi\overline{M} + \cdots \\ \cdots fM + \int_{\phi^*}^{\overline{\phi}} nf^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF\left(y\right)g\left(\phi'\right)\overline{M}d\phi' \end{array} = L, \quad (19)$$

which states that the total value of wages paid by firms to production and nonproduction workers (left) equals total labor income in the economy, where the expression for the latter uses the numeraire assumption. As in the closed economy case, the conditions derived in this section are not only necessary, but also sufficient for an equilibrium. This characterization of the equilibrium is summarized in lemma 3 in the appendix, which can be easily proved adapting the arguments in the proof of lemma 1.

I conclude this section with a summary of the qualitative properties of the equilibrium in the open economy. In equilibrium, more-productive firms employ *production* workers of higher ability and pay them higher wages. The stochastic specification of fixed export costs yields an imperfect positive correlation between firms' productivity, average workforce ability, size and export status, which is consistent with the empirical evidence documented in Bernard and Jensen (1995) and Bernard, Eaton, Jensen, and Kortum (2003).

4.2 Existence and Uniqueness of the Equilibrium

I start this section by studying the existence and uniqueness of solutions to the nonlinear, two-point BVPs characterizing the equilibrium in the closed and open economies. In contrast to the cases of initial value problems (IVPs) and linear BVPs, for which there is a standard theory that provides fairly general results under relatively mild restrictions on the *data* of the problem, such a study is not trivial in the case of nonlinear BVPs for two reasons.[21] First, there is no unified theory that can be applied to study these issues for an arbitrary problem. Because of the complexity of the subject, the mathematical literature has typically focused on particular cases of the problem, leading to a multitude of theoretical approaches tailored to these cases.[22] Second, most results in the literature are based on restrictive and not-easily-verifiable assumptions, while those results based on less restrictive assumptions, resembling those used in the standard theory of IVPs, have a local flavor.[23] Despite these difficulties, several studies in the trade literature that use assignment models and arrive to characterizations of the equilibrium involving a BVP similar to those above, simply assume or state without proof the existence and uniqueness of the solution. In this section, I fill this gap in the trade literature by presenting existence and uniqueness results for a nonlinear BVP that encompasses the two BVPs considered above and others in the literature.[24]

For any $\phi_0, \phi_1 \in [\underline{\phi}, \overline{\phi}]$ and $s_0, s_1 \in [\underline{s}, \overline{s}]$, with $\phi_0 < \phi_1$ and $s_0 < s_1$, I consider the nonlinear, two-point BVP (20), comprising the system of differential equations (20a)-(20c) and the boundary conditions (20d),

$$
\begin{aligned}
z_\phi(\phi) &= -z(\phi)\frac{\partial \ln A(\Gamma(\phi),\phi)}{\partial\phi}, && \text{(20a)}\\
x_\phi(\phi) &= (\sigma-1)\,x(\phi)\frac{\partial \ln A(\Gamma(\phi),\phi)}{\partial\phi}, && \text{(20b)}\\
\Gamma_\phi(\phi) &= \frac{x(\phi)\left[1+F\left(K_0 x(\phi)\right)K_1\right]\alpha(\phi) g(\phi)}{A(\Gamma(\phi),\phi)\,V(\Gamma(\phi))\,z(\phi)}, && \text{(20c)}\\
x(\phi) &= 1,\ \Gamma(\phi_0)=s_0,\ \Gamma(\phi_1)=s_1, && \text{(20d)}
\end{aligned}
$$

where $\alpha(\phi)$ is a strictly positive continuous function, $\alpha : [\underline{\phi}, \overline{\phi}] \to \mathbb{R}_{++}$, K_0 and K_1 are nonnegative

[21] For a discussion of standard existence and uniqueness theory for IVPs see Agarwal and O'Regan (2008a), which also covers basic results for linear BVPs. For a more comprehensive treatment of linear BVPs see Stakgold (1998) and Agarwal and O'Regan (2008b).

[22] Bernfeld and Lakshmikantham (1974) survey the most common problems and theoretical approaches considered in the literature. See Kiguradze (1988) for some results for the general two-point BVP.

[23] Bailey, Shampine, and Waltman (1968) present several existence and uniqueness results for nonlinear BVPs using Piccard's Iteration method when the functions involved satisfy certain Lipschitzian conditions. In all cases, the interval over which the solution is defined has to be sufficiently small.

[24] The general BVP considered in this section encompasses those in Costinot and Vogel (2010), Sampson (2014), Somale (2015), Grossman, Helpman, and Kircher (2017).

constants and $\{A, g, V, F\}$ are the functions defined earlier.

The general BVP defined above nests the BVPs corresponding to the closed and open economies, as the latter can be obtained as particular parametrizations of the former. If we set $K_0 = (f/f_x)\tau^{1-\sigma}$, $K_1 = n\tau^{1-\sigma}$, $\phi_0 = \phi^*$, $\phi_1 = \overline{\phi}$ and $\alpha(\phi) = 1$ for all $\phi \in [\underline{\phi}, \overline{\phi}]$, the resulting BVP is equivalent to the BVP of the open economy, in the sense that any solution to one of these two BVPs can be used to construct a solution to the other. To see this, let $\{z, x, \Gamma\}$ be a solution to the BVP (20) parametrized as above. If we define $r^d(\phi) \equiv \sigma f x(\phi)$, $p(\phi) \equiv z(\phi)\sigma f\overline{M}/[L - fM - \int_{\phi^*}^{\overline{\phi}} nf_x \int_0^{fx(\phi')\tau^{1-\sigma}/f_x} ydF(y)\, g(\phi')\, \overline{M} d\phi']$ and $H = \Gamma$, then $\{p, r^d, H\}$ is a solution to the BVP of the open economy. A similar argument shows that any solution to the BVP of the open economy can be used to construct a solution to this particular parametrization of BVP (20). Finally, if we set $K_1 = 0$ in the parametrization above, the resulting BVP is equivalent to the BVP of the closed economy defined in lemma 1.ii.

Lemma 2 states some important results about the general BVP (20).

Lemma 2 *If the right-hand side of equations (20a)-(20c) are locally Lipschitz continuous with respect to* $\{z, x, \Gamma\}$*, then there is a unique continuously differentiable solution to the BVP (20) for any* $\phi_0, \phi_1 \in [\underline{\phi}, \overline{\phi}]$ *and* $s_0, s_1 \in [\underline{s}, \overline{s}]$*, with* $\phi_0 < \phi_1$ *and* $s_0 < s_1$*. As a function of* (ϕ_0, s_0)*, the solution to the BVP,* $\{z(.;\phi_0, s_0), x(.;\phi_0, s_0), \Gamma(.;\phi_0, s_0)\}$*, satisfies the following conditions,*
(i) (no crossing) If $K_1 = 0$ *and* Γ^{-1} *is the inverse of* Γ*, then* $s_0^a < s_0^b$ *implies* $\Gamma(\phi;\phi_0, s_0^a) < \Gamma(\phi;\phi_0, s_0^b)$ *on* $[\phi_0, \phi_1)$*, while* $\phi_0^a > \phi_0^b$ *implies* $\Gamma^{-1}(s;\phi_0^a, s_0) > \Gamma^{-1}(s;\phi_0^b, s_0)$ *on* $[s_0, s_1)$*.*
(ii) $\phi_0^a > \phi_0^b$ *implies* $x(\phi;\phi_0^a, s_0) < x(\phi;\phi_0^b, s_0)$ *on* $[\phi_0^a, \phi_1]$*.*

I present a brief outline of the proof of the last lemma below, relegating the details to the appendix. To prove existence, I follow O'Regan (2013) and recast the BVP as a fixed point problem. In particular, I show that a triplet $\{z, x, \Gamma\}$ solves BVP (20) if and only if Γ is a fixed point of some compact functional, Ψ, defined over a convex and closed set K, $\Psi(\Gamma) = \Gamma$. Then, a direct application of Schauder fixed point theorem yields the existence result. The uniqueness of the solution is established as a consequence of the particular structure of the problem and the strict log-supermodularity of A. Lemma 1.i is obtained as a corollary of the uniqueness result. For $K_1 = 0$ (closed economy), lemma 1.ii immediately follows from the previous no-crossing result, (20b) and the log-supermodularity of A. However, this argument cannot be extended to the case $K_1 > 0$ (open economy), as the no-crossing property no longer holds. In the appendix, I present a slightly longer argument that is valid for the general case $K_1 \geq 0$, which also establishes the result as a consequence of the strict log-supermodularity of A.

An important corollary of the discussion so far is that, for a given activity cutoff ϕ^*, the functions r^d and H that solve the BVPs of the closed and open economies do not depend on the mass of firms, $\overline{M}$, nor the mass of production workers.[25] This feature of the solution follows from the uniqueness result in lemma 2, equation (20c) and the correspondence between said BVPs and BVP (20) described above. In fact, the mass of firms and the mass of production workers affect only the level of the solution function p. This result will prove useful in the analysis of the free-entry model in section 6.

[25] The mass of production workers in the closed and open economies are given by the term in brackets in the denominator of the right-hand side of equations (13) and (18), respectively.

As the BVP of the open economy has a unique solution conditional on the activity cutoff ϕ^*, then there exists a unique equilibrium of the open economy if and only if there is a unique value of ϕ^* that solves equation (19). Given the correspondence between the open-economy BVP and the general BVP (20), lemma 2.ii implies that $r^d(\phi)$ is strictly decreasing in the activity cutoff ϕ^*, making the left-hand side of (19) strictly decreasing in the value of ϕ^*. As the right-hand side of (19) does not depend on ϕ^*, there is a unique solution to (19) if the size of the market, as captured by L, is not too large.[26] A similar argument shows that there is a unique equilibrium in the closed economy. I summarize this discussion in the next proposition, which also establishes the (constrained) efficiency of the equilibrium.

Proposition 1 *Let $\{\underline{p}, \underline{r}^d, \underline{H}\}$ and $\{\underline{p}^a, \underline{r}^{d,a}, \underline{H}^a\}$ be, respectively, the solution to the BVPs characterizing the open- and closed-economy equilbria with $\phi^* = \underline{\phi}$. In addition, let $\beta(r^d, \phi^*)$ and $\beta^a(r^d, \phi^*)$ denote the functions defined by the left-hand sides of equations (19) and (14), respectively, in terms of ϕ^* and r^d.*
(i) For $\beta(\underline{r}^d, \underline{\phi}) > L$, there is a unique no-free-entry equilibrium of the open economy.
(ii) For $\beta^a(\underline{r}^d, \underline{\phi}) > L$, there is a unique no-free-entry equilibrium of the closed economy.
In addition, the equilibrium of the closed economy is efficient, while that of the open economy is efficient when $f \leq f_x \tau^{1-\sigma}$, and constrained efficient when $f > f_x \tau^{1-\sigma}$.

5 No Free Entry, Trade and Wage Inequality

In this section, I study the effects of higher trade openness on wage inequality in the no-free-entry model described above. In the model, a decline in trade frictions induces a reallocation of production and employment across firms with heterogenous skill demand, affecting the aggregate relative demand for skills and the relative wages in the economy. In the analysis, I decompose these effects into the contributions of each of the three channels defined in the introduction—the selection-into-activity, intensive-margin and extensive-margin channels.

Being a sufficient statistic for the dispersion of wages in the model, the matching function takes center stage in the subsequent analysis, as any result about wage inequality in this framework is a statement about the impact on the matching function of the shock under consideration. Lemma 4 in the appendix collects several results related to the general BVP in (20) that are instrumental to the analysis. In particular, this lemma characterizes the dependence of the solution function Γ (and some functionals of Γ) on the parameters of the problem.

5.1 Autarky vs. Trade

The first instance of higher trade openness that I consider is the case of an initially autarkic economy that opens up to trade. I start this section with one of the main results of the paper, Proposition 2, which states that opening to trade leads to a pervasive increase in wage inequality.

[26] If L is too large relative to the mass of firms, $\overline{M}$, then there is no equilibrium featuring selection into activity as all firms make postive profits.

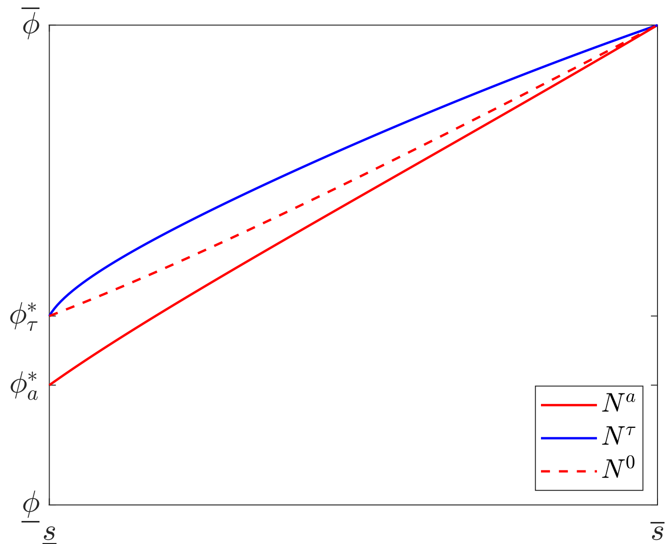
Proposition 2 *Let $\{\phi_a^*, N^a\}$ and $\{\phi_\tau^*, N^\tau\}$ be the activity cutoffs and matching functions corresponding to the no-free-entry equilibrium of the closed and open economies, respectively. Then the following conditions hold:*
(i) $\phi_\tau^* > \phi_a^*$ and $N^\tau(s) > N^a(s)$ for all $s \in [\underline{s}, \overline{s})$, so inequality is pervasively higher in the open economy.
(ii) The selection-into-activity and extensive-margin channels lead to pervasively higher inequality (intensive-margin channel not operational).

The first result in the last proposition, $\phi_\tau^* > \phi_a^*$, states that the selection-into-activity effects of trade highlighted in Melitz (2003) always hold in the no-free-entry model of this paper—i.e., trade induces the least productive firms to exit the market. Although somewhat trivial in homogenous-workers models a-lá Melitz/Channey, this result is not immediate in the current framework. For example, in an homogenous-workers version of the no-free-entry model above, assuming that firms with productivity ϕ_a^* are still active after the economy starts trading results in unchanged domestic revenues and labor costs. With aggregate labor costs pinned down by an equilibrium condition, this observation, together with positive export labor costs, implies that a higher activity cutoff is required in the open economy. In contrast, making the same assumption in the heterogeneous-worker framework above leads to lower domestic revenues and labor costs, so establishing the result requires proving that the decline in the latter is more than offset by the new labor costs of exporting (variable and fixed). I do so in the appendix by showing that total wages paid to *production* workers necessarily increase if the activity cutoff remains unchanged, which together with the presence of fixed export labor costs, leads to a rise in the the total wages paid by firms. With total wages pinned down by the numeraire assumption, condition (19), a higher activity cutoff is required in the open economy.[27]

To gain more insight into the effects of opening to trade on wage inequality, I decompose the overall effect into the three channels defined earlier. First of all, note that the intensive-margin channel is not operational in this case, as there were no exporters before the economy started to trade. The selection-into-activity channel captures the impact on wage inequality of the trade-induced increase in the activity cutoff, excluding the impact of changes in the set of exporters. To isolate the effect of this channel, I contrast the matching function of the closed economy with that of an ancillary autarkic economy that differs from the former only in that its activity cutoff is given by that of the open economy. That is, the equilibria of the closed and ancillary economies are characterized by the BVP in lemma 1.ii with $\phi^* = \phi_a^*$ and $\phi^* = \phi_\tau^*$, respectively. The typical situation is depicted in figure 1, where the solid and dashed red lines are, respectively, the matching functions of the closed (N^a) and ancillary (N^0) economies. The no-crossing result in lemma 2.i. implies that the latter lies strictly above the former on $[\underline{s}, \overline{s})$ as shown in the figure. Intuitively, as the firms with productivity in the range $[\phi_a^*, \phi_\tau^*)$ become inactive, the aggregate demand for workers with skills in the range $[\underline{s}, N^a(\phi_\tau^*))$ drops to zero barring any change in the wage schedule. Per the labor market clearing condition, these workers must be reallocated among the firms that remain active, requiring a decline in their relative wages.

[27] As explained earlier, the left-hand side of (19) is strictly decreasing in the activity cutoff.

Figure 1: Opening to Trade and the Matching Function



Note: The solid red and blue lines represent, respectively, the matching functions of the closed (N^a) and open (N^τ) economies. The dashed red line depicts the matching function of the ancillary autarkic economy (N^0) described in the text. The differences between N^a and N^0 and between N^0 and N^τ capture the impact of the selection-into-activity and extensive-margin channels, respectively.

The extensive-margin channel reflects the impact on wage inequality of the increased labor demand by new exporters as they expand their production to serve the foreign market, excluding the effects of changes in the activity cutoff. Put another way, this channel captures the effects of replacing $[1 + F\left(r^d(\phi)\,\tau^{1-\sigma}/\sigma f^x\right) n\tau^{1-\sigma}]$ with 1 in the BVP of the open economy, precisely what the difference between the matching functions of the ancillary (N^0) and open (N^τ) economies in figure 1 captures, with the latter shown in blue. To see why N^τ necessarily lies above N^0 as depicted in the figure, suppose for a moment that the wages of the ancillary economy also prevail in the open economy. In this case, firms of a given productivity level demand the same skill type of workers in both economies, with exporters in the open economy demanding more labor than nonexporters due to the foreign demand they face. If the fraction of exporters was constant across productivity levels, this additional export-driven labor demand would affect all skill levels proportionally, leaving unchanged the overall relative demand for skills in the economy. However, as the fraction of exporters in the model increases with firms' productivity, this additional export-driven labor demand is tilted towards more-able workers, resulting in an excess demand for this type of labor. As such, market clearing requires higher relative wages for more-skilled workers in the open economy.[28]

I conclude this section with a discussion of the impact of trade on the *level* of real wages. Although trade always raises the average real wage, the least-skilled workers in the economy may see their real wage decline. The pricing rule (5) and the zero profit condition (7) imply that the aggregate price indices of

[28] Formally, in the appendix I show that the BVPs of the ancillary and open economies can be conceived as particular parameterizations of the general BVP (20) with $K_1 = 0$ that differ only in the parameter function $\alpha(\phi)$, which is constant in the former and increasing in the latter. The result then follows from a direct application of lemma 4.i in the appendix.

the closed (P^a) and open (P^τ) economies satisfy

$$\left(P^i\right)^\sigma = \frac{\sigma f}{U^i}\left[\frac{\sigma}{(\sigma-1)}\frac{w^i\left(\underline{s}\right)}{A\left(\underline{s},\phi_i^*\right)}\right]^{\sigma-1} \quad \text{for } i = a,\tau, \tag{21}$$

where U^i is the aggregate real expenditure/income in the economy. Per the efficiency result in proposition 1, real income is higher in the open economy, $U^\tau > U^a$.[29] In addition, proposition 2.i, together with the numeraire assumption ($\overline{w}^i = 1$), implies that the open economy exhibits a higher activity cutoff, $\phi_\tau^* > \phi_a^*$, and a lower wage for the least-able workers, $w^\tau\left(\underline{s}\right) < w^a\left(\underline{s}\right)$. Accordingly, $P^\tau < P^a$, so the average real wage, $\overline{w}/P$, is higher in the open economy.

Finally, recalling that $U^i = E^i/P^i$, equation (21) can be rearranged to get the an expression for the real wage of the least-able workers, $\frac{w^i(\underline{s})}{P^i} = \frac{(\sigma-1)}{\sigma}A\left(\underline{s},\phi_i^*\right)\left[E^i/\sigma f\right]^{\frac{1}{\sigma-1}}$. This expression implies that opening to trade necessarily improves the real wage of these workers when it induces a rise in aggregate expenditure/income. However, in some parameterizations of the model, opening to trade can induce a decline in the real wage of the poorest workers, as the drop in aggregate income more than offsets the boost from working at a more productive employer (higher activity cutoff).

5.2 Trade Liberalization

Although the preceding analysis sheds light into the effects of higher trade openness on wage inequality, very few, if any, of the countries in the world operate in autarky. For this reason, in this section I study the effects on wage inequality of a trade liberalization, defined as a decline in the variable trade costs faced by an economy that already participates in international trade. As described in proposition 3, I find that these effects may differ from those described in the previous section. In particular, although a trade liberalization necessarily raises wage inequality among the least-skilled workers in the economy, wage inequality may decline elsewhere in the wage the distribution.

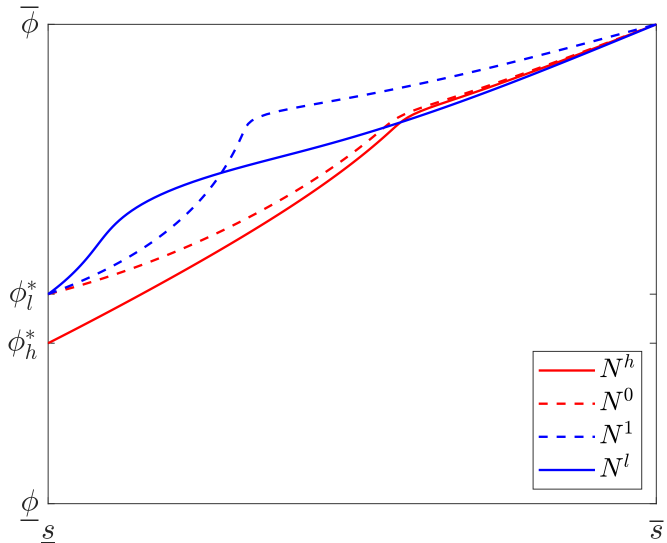
Proposition 3 *Consider a trade liberalization that reduces variable trade costs from τ_h to τ_l, and let $\left\{\phi_h^*, N^h\right\}$ and $\left\{\phi_l^*, N^l\right\}$ represent, respectively, the pre- and post-liberalization activity cutoffs and matching functions. Then, the following conditions hold:*
(i) $\phi_l^* > \phi_h^*$, so a trade liberalization raises wage inequality among the least-skilled workers.
(ii) The selection-into-activity and intensive-margin channels lead to pervasively higher inequality, while the effect of the extensive-margin channel is ambiguous.
(iii) Let $\eta_0^F(t,\lambda) \equiv \frac{F_y(t\lambda)\lambda}{[1+F(t\lambda)k]}$, $\eta_1^F(t,\lambda) \equiv \frac{F_y(t\lambda)\lambda^2}{[1+F(t\lambda)k]}$, and $\bar{t} \equiv \frac{r^{d,a}(\overline{\phi})}{\sigma f^x}$, where $r^{d,a}(\overline{\phi})$ is the autarky revenue function. If the functions η_0^F and η_1^F are, respectively, strictly decreasing and strictly increasing in λ for $\lambda \geq 1$, $k \in (0,n)$ and $t \in (\underline{y},\bar{t})$, then a trade liberalization raises wage inequality pervasively.

The first result of the proposition states that, as in the Melitz/Channey models, a trade liberalization always leads to the exit of the least productive of firms from the market, $\phi_l^* > \phi_h^*$. The general line

[29] Note that the closed economy allocation is available to the planner of the open economy, so a simple revealed-preference argument yields $U^\tau > U^a$.

of argument used in the proof of proposition 2.i. can be applied here as well. If the activity cutoff remains unchanged after the decline in trade costs, then total wages paid to production and nonproduction workers necessarily increase. With total wages pinned down by condition (19), the activity cutoff must be higher after the liberalization. This result and the continuity of the matching functions imply that $N^l(s) > N^h(s)$ on some interval of the form $[\underline{s}, s')$, which is equivalent to the second part of the claim in proposition 3.i.[30]

Figure 2: Trade Liberalization and the Matching Function



Note: The solid red and blue lines represent, respectively, the pre- (N^h) and post-liberalization (N^l) matching functions described in Proposition 3. The dashed red (N^0) and dashed blue lines (N^1) depict the matching functions of the ancillary economies described in the text. The effects of the selection-into-activity, intensive-margin, and extensive-margin channels on the matching function are captured, respectively, by the differences between the pairs $\{N^h, N^0\}$, $\{N^0, N^1\}$, and $\{N^1, N^l\}$.

As before, the overall impact of a trade liberalization on wage inequality can be decomposed into the three channels defined earlier. The selection-into-activity channel captures the changes in wage dispersion associated with the rise in the activity cutoff, excluding the impact of changes in the labor demand of incumbent exporters and of changes in the set of exporters. To isolate the effect of this channel, I contrast the matching function of the open economy before the liberalization, N^h, with that of an ancillary open economy, N^0, that differs from the former only in that its activity cutoff is given by that prevailing after the liberalization, ϕ_l^*. That is, as I explain in more detail in the appendix, the BVPs associated with N^h and N^0 can be conceived as parameterization of the general BVP (20), with $K_1 = 0$ and $\alpha^h(\phi) \equiv [1 + F\left(r^{d,h}(\phi)\,\tau_h^{1-\sigma}/\sigma f^x\right) n\tau_h^{1-\sigma}]$, that differ only in their boundary conditions.[31] Accordingly, the no crossing result in lemma 2.i. implies that N^0 lies strictly above N^h on $[\underline{s}, \overline{s})$ as depicted by the dashed and solid red lines in figure 2. The intuition for the effects of this channel are the same as before—

[30] Of note, establishing the consequences of an unchanged activity cutoff is more complicated in the case of a trade liberalization, as multiple crossings of relevant matching functions cannot be ruled out. In this case, the formal argument is based on the results in lemma 4.iv-v.

[31] $r^{d,h}$ is the domestic revenue function of the open economy with variable trade costs τ_h.

i.e., the exit of the least-productive firms from the market reduces the relative demand for less-skilled workers, pushing down their relative wages.

The intensive-margin channel captures the impact on wage inequality of the liberalization-induced rise in the labor demand of incumbent exporters. I isolate this channel by contrasting the matching function N^0 with that of a second ancillary open economy, N^1, with the same set of exporters and active firms, but with variable trade costs given by τ_l. That is, N^1 is obtained by replacing the parameter function $\alpha^h(\phi)$ with $\alpha^1(\phi) \equiv [1 + F\left(r^{d,h}(\phi)\,\tau_h^{1-\sigma}/\sigma f^x\right) n\tau_l^{1-\sigma}]$ in the BVP associated with N^0. As shown by the dashed blue and red lines in figure 2, N^1 necessarily lies above N^0 on $(\underline{s}, \overline{s})$ for the same reasons laid out in the discussion of the extensive-margin channel in proposition 2. If these ancillary economies shared the same wage schedule, then firms of a given productivity level would demand the same worker type in both economies, with the N^1-economy exhibiting a larger labor demand from exporters (lower trade costs). As the (common) fraction of exporters in these economies is increasing in firms's productivity, this additional export-driven labor demand in the N^1-economy results in a higher relative demand for more-skilled workers, which is inconsistent with labor market clearing. Accordingly, the wages of these workers must be higher in the N^1-economy.[32]

The extensive-margin channel captures the impact on relative wages of allowing the fraction of exporters to adjust—i.e., the effects on wages of replacing $\alpha^1(\phi)$ with $[1 + F\left(r^{d,l}(\phi)\,\tau_l^{1-\sigma}/\sigma f^x\right) n\tau_l^{1-\sigma}]$ in the BVP associated with N^1. Little can be said about these effects without making additional assumptions about the primitives of the model. In figure 2, which illustrates only one of the many possibilities, the impact of this channel is given by the difference between N^1 and N^l, the dashed and solid blue lines, respectively. In this example, the weight of some middle-productivity firms among exporters in the post-liberalization economy is larger than in the ancillary N^1-economy. Then, the change in the set of exporters drives up the relative demand for some middle-skill workers, pushing up their wages relative to those of workers with lower and higher skill levels. That said, the impact of this channel could take other forms depending on the CDF of fixed export costs, F, including a pervasive rise and a pervasive decline in wage inequality. Moreover, the effects of this channel can be strong enough to offset the impact of the other two channels in some parts of the wage distribution, as shown by the crossing of N^h and N^l in figure 2.

Proposition 3.iii presents a set of sufficient conditions on the CDF of fixed exports costs, F, that guarantee that a trade liberalization always leads to a pervasive rise in wage inequality. When the condition on the function η_1^F is satisfied, reducing variable trade costs while keeping the activity cutoff unchanged in the BVP of the open economy (that allows the set of exporters to change) always leads to pervasively higher wage inequality. In addition, when the condition on η_0^F is satisfied, increasing the activity cutoff while keeping variable trade costs constant in said BVP also leads to a pervasive rise in wage dispersion. Accordingly, when both conditions are met, wage inequality increases pervasively following a liberalization, as the effect on relative wages of changes in the set of exporters (extensive-margin channel) never offsets the combined impact of the selection-into-activity and intensive-margin channels. Although these restrictions on F may appear very restrictive to some readers, one should bear in mind that they

[32] The result follows from a direct application of lemma 4.i in the appendix, with α^1 taking the role of α^a in the lemma.

are sufficient conditions under all parameterizations of the model.[33]

Regarding the impact of a trade liberalization on the *level of wages*, the analysis and conclusions of the previous section also apply to this case. A liberalization increases real income and average real wages, but the least productive workers in the economy could see their real wage decline in some parameterizations of the model.

5.3 Trade and Wage Dispersion in Other Frameworks

The three-channel decomposition of the effects of higher trade openness on wage inequality described above can be a useful tool to analyze differences in the implications of alternative frameworks in the literature. For illustration purposes, I compare the effects of opening to trade on wage inequality in the no-free-entry model in this paper with those in Helpman, Itskhoki, and Redding (2010), henceforth HIR. In the HIR model, firms screen workers to improve the composition of their labor forces as worker ability is not directly observable. As larger firms have higher returns from screening, they do so more intensively and have workforces of higher average ability than smaller firms. This mechanism generates a wage-size premium, implying that both productivity and exporting positively affect the average wages paid by a firm.

In the HIR model, wage inequality increases after an economy opens to trade only when there is selection into exporting (only some firms export), but is unchanged when all firms become exporters. In terms of the three channels defined earlier, the selection-into-activity channel is not operational in the HIR model, as changes in the activity cutoff do not modify the relative size of firms. In addition, the extensive-margin channel affects wage inequality only when it changes the relative size of firms in the economy— i.e., only when some but not all firms export. In contrast, trade always leads to higher wage inequality in the no-free-entry model of this paper. Although trade may not affect wage inequality through the extensive-margin channel if all firms export (as in HIR), it always drives up wage dispersion through the selection–into-activity channel.

6 The Free-Entry Model

In the model outlined above, the mass of firms in the industry is fixed at an exogenous level. Although this assumption may be a good approximation to the firm-entry dynamics in the short-run, it does not capture the change in the number of firms through endogenous entry and exit over time. In this section, I relax this assumption by allowing firms to enter the industry for a cost, making the mass of firms in the industry, $\overline{M}$, an additional endogenous variable. Specifically, I assume that there is an unbounded pool of prospective firms that can enter the industry by incurring a fixed entry cost of $f^e V(s)$ units of each skill $s \in S$. Accordingly, the aggregate expenditure on entry costs is $\overline{M} f^e$ when a mass $\overline{M}$ of firms enters

[33] For a Pareto distribution, the condition on η_0^F is always satisfied, while that on η_1^F is satisfied when the shape parameter is small enough. Moroever, a sufficiently small shape parameter typically precludes the crossing of the matching function even when the condition on η_1^F is not satisfied.

the industry. Upon entry, firms obtain their productivity as independent draws from the distribution G, as explained in section 2.2. All the other primitives of the model remain unchanged.

The new assumptions above do not affect the basic structure of the model described in section 2, so equations (1)-(5) continue to hold. Conditional on the mass of firms, $\overline{M}$, the equilibrium analysis in section 4 applies almost unchanged to the free-entry model, with the caveat that equilibrium conditions now reflect the labor demand derived from the presence of fixed entry costs—i.e., L must be replaced with $L - f^e\overline{M}$ throughout the analysis. The new free-entry assumption implies that, in equilibrium, prospective entrants must be indifferent between entering and not entering the industry. Accordingly, expected profits from entering the industry must equal the cost of entry, $[1 - G(\phi^*)]\left[\overline{\pi}^d + \overline{\pi}^x\right] = f^e$, where $\overline{\pi}^d$ and $\overline{\pi}^x$ are, respectively, the average domestic and export profits among active firms.[34] Per the optimal pricing rule, this *free-entry* condition can be written as

$$\int_{\phi^*}^{\overline{\phi}} \left[\frac{r^d(\phi)}{\sigma} - f\right] g(\phi)\, d\phi + \int_{\phi^*}^{\overline{\phi}} \int_0^{\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}} n \left[\frac{r^d(\phi)\,\tau^{1-\sigma}}{\sigma} - f^x y\right] dF(y)\, g(\phi)\, d\phi = f^e. \tag{22}$$

The last equation completes the description of the open-economy equilibrium in the free-entry model, prompting a definition analogous to that in definition 1.

The free-entry equilibrium of the open economy is subject to a characterization analogous to that given in section 4.1 for the no-free-entry model. In particular, given the activity cutoff, ϕ^*, the price, domestic-revenue and inverse-matching functions, $\{p, r^d, H\}$, solve a BVP that differs from that of the no-free-entry model in lemma 3.iii. only in that L is replaced by $L - f^e\overline{M}$ in the equation defining the slope of the inverse-matching function. Moreover, the discussion in section 4.2 implies that conditional on ϕ^*, the BVPs of the no-free-entry and free-entry models have the same parameterization in terms of the general BVP (20), so they share the same solution functions r^d and H. The equilibrium value for ϕ^* is pinned down by the free entry condition (22).[35]

The observations above have important implications. First, all the conclusions reached in section 4.2 about the dependence of $\{r^d, H\}$ on the activity cutoff ϕ^* continue to hold in the free-entry model. Accordingly, many results, such as the existence and uniqueness of the equilibrium in the free-entry model, can be derived in a similar way.[36] Second, the only relevant difference between the no-free-entry and free-entry models regarding the determination of the equilibrium matching function is given by the equations that pins down the activity cutoff in these models, equations (19) and (22), respectively. In the remainder of this section I explore how this difference affects the impact of increased trade openness on wage inequality.

[34] Note that $\overline{\pi}^x$ is not the average export profits among exporters, but among all active firms.

[35] This is the case because ϕ^* and r^d are the only endogenous variables appearing in equation (22). Note that using the analog of equation (19) for the free-entry model to determine the activity cutoff ϕ^* would only give us ϕ^* as a function of the endogenous mass of firms $\overline{M}$.

[36] As $r^d(\phi)$ depends negatively on the activity cutoff, the left-hand side of equation (22) is strictly decreasing in ϕ^*, implying that there is unique free-entry equilibrium if entry costs are not too high.

6.1 Autarky vs. Trade in the Free-entry Model

Unlike the case of the no-free-entry model, an increase in trade openes may lead to a rise or fall in the activity cutoff in the free-entry model, with ambiguous effects on the wage distribution through the selection-into-activity channel. As formally stated in proposition 4, this additional source of ambiguity in the free-entry model implies that opening to trade can lead a pervasive rise in wage inequality or a wage polarization.

Proposition 4 *Let $\{\phi_a^*, N^a\}$ and $\{\phi_\tau^*, N^\tau\}$ be the activity cutoffs and matching functions corresponding to the free-entry equilibrium of the closed and open economies, respectively. Then ϕ_τ^* could be lower or higher than ϕ_a^* depending on the model's parameters.*
(i) If $\phi_\tau^* \geq \phi_a^*$, then $N^\tau(s) > N^a(s)$ on $s \in (\underline{s}, \overline{s})$, so opening to trade leads to pervasively higher wage inequality. The selection-into-activity channel leads to a pervasive rise (no change) in wage inequality if $\phi_\tau^* > (=) \phi_a^*$. The extensive-margin channel always leads to a pervasive rise in wage inequality.
(ii) If $\phi_\tau^* < \phi_a^*$, then $N^\tau(s)$ and $N^a(s)$ intersect exactly once on $(\underline{s}, \overline{s})$, so opening to trade leads to wage polarization. The selection-into-activity and extensive-margin channels lead, respectively, to pervasively lower and pervasively higher wage inequality.

I start the discussion of propostition 4 by analyzing why opening to trade may lead to a decline in the activity cuoff in the free-entry model. As this theoretical possibility is not present in the no-free-entry model in this paper nor in standard free-entry models with homogeneous workers, such as Melitz (2003), I discuss the differences between these two frameworks and the free-entry model in this paper that allow for this additional possibility in the latter.

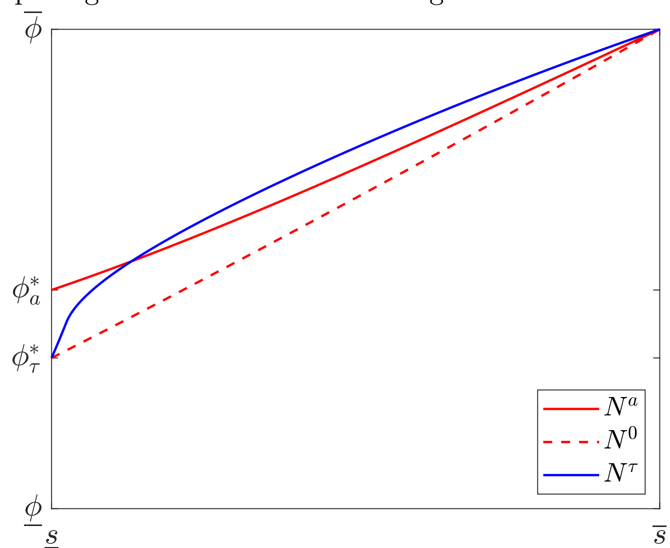
Opening to trade may have different qualitative effects on the activity cutoff in the no-free-entry and free-entry models of this paper, reflecting the different equilibrium conditions that determine this cutoff in these models. These differences are better understood by comparing the impact that trade has on these equilibrium conditions when the set of active firms and the revenue of the least-productive ones are assumed to remain unchanged, $r^d(\phi_a^*) = \sigma f$. As discussed in section 5, in this scenario, trade leads to a rise in the implied total wages paid to production and nonproduction workers, as total firms' revenue and fixed export costs increase. Accordingly, equation (19) implies that a higher activity cutoff is required in the open economy of the no-free-entry model. In contrast, in the free-entry model, total firms' revenue and fixed export costs enter with opposite signs on the left-hand side of the free-entry condition (22), with an ambiguous net effect, so a lower activity cutoff may be required in the open economy.

Relative to standard free-entry models with homogeneous workers, a trade-induced decline in the activity cutoff is possible in the free-entry model because of the endogenous changes in the matching of heterogeneous workers to firms.[37] As before, it is instructive to compare the impact that trade has on the free-entry condition in these models when the set of active firms and the revenue of the least-productive ones are assumed to remain unchanged. In such a scenario, trade increases export profits from zero (in autarky) to some strictly positive number in both models. With domestic profits remaining unchanged

[37] The stochastic modeling of fixed costs is another difference between the free-entry model in this paper and standard Melitz-type models. However, said difference alone cannot produce a trade-induced declined in the activity cutoff.

in the homogeneous-workers model (before adjusting the activity cutoff), average/expected profits necessarily increase, so the free-entry condition requires a higher activity cutoff in the open economy. In contrast, in the free-entry model of this paper, trade may lead to a decline in aggregate profits due to changes in the matching function. Specifically, as the matching function N shifts up (H shifts down) in the scenario considered, domestic revenues and profits decline for firms with productivity above ϕ_a^*. For some parameter values, the decline in aggregate domestic profits more than offsets the rise in export profits, so the free-entry condition (22) requires a lower activity cutoff in the open economy.

Figure 3: Opening toTrade and the Matching Function in Free-entry Model



Note: The solid red and blue lines represent, respectively, the matching functions of the closed (N^a) and open (N^τ) economies. The dashed red line depicts the matching function of the ancillary autarkic economy (N^0) described in the text. The differences between N^a and N^0 and between N^0 and N^τ capture the impact of the selection-into-activity and extensive-margin channels, respectively. The figure depicts the case in which trade induces a decline in the activity cutoff.

Per proposition 4, conditional on its impact on the activity cutoff, trade has a unique qualitative effect on the dispersion of wages, with an unambiguous effect through the selection-into-activity and extensive-margin channels. The case in proposition 4.i, $\phi_\tau^* \geq \phi_a^*$, is essentially the same situation considered in section 5.1 for the no-free-entry model. If $\phi_\tau^* > \phi_a^*$, then the situation is identical to that depicted in figure 1, so the corresponding analysis applies here as well. When $\phi_\tau^* = \phi_a^*$, the only difference is that the selection-into-activity channel has no effect on wage dispersion.

The case in proposition 4.ii, $\phi_\tau^* < \phi_a^*$, requires some additional explanation. As I discuss in the appendix, the matching function of the open economy, N^τ, cannot remain completely below that of the closed economy, N^a, on $[\underline{s}, \overline{s})$. Otherwise, per lemma 2.ii, expected domestic profits in the open economy would be strictly higher than in autarky, implying a violation of the free-entry condition (22). Then, N^τ and N^a must intersect at least once on $(\underline{s}, \overline{s})$. Moreover, adapting the analysis of the extensive-margin channel in section 5.1 to assess the relative position of N^a and N^τ to the right of the first intersection, it can be shown that N^a must remain below N^τ there, so the matching functions must intersect exactly once on $(\underline{s}, \overline{s})$.[38] The situation is depicted in figure 3, where the solid red and blue lines represent N^a and N^τ, respectively. As before, the dashed red line is the matching function of an ancillary autarkic economy, N^0, that is obtained by changing the activity cutoff in the BVP corresponding to N^a from ϕ_a^* to ϕ_τ^*. As discussed in section 5.1, the effects of trade on wage inequality through the selection-into-activity and extensive-margin channels are captured, respectively, by the difference between the pairs $\{N^a, N^0\}$ and $\{N^0, N^\tau\}$. While the selection-into-activity channel pervasively reduces wage inequality, the extensive-margin channel pervasively increases it, with the former channel dominating to the left of the interior intersection point of N^a and N^τ, and the latter dominating to the right. As a result, workers with skill level corresponding to this (interior) intersection point see their wages decline relative to those of all other workers— i.e., opening to trade leads to wage polarization.

Turning to the effects of trade on the *level of real wages*, the results obtained for the no-free-entry model generally go through. First, the average real wage is always higher in the open economy. As before, the result follows from the (constrained) efficiency of the equilibrium. Second, opening to trade may induce a decline in the real wage of the least-skilled workers in the economy, although in the free-entry model this possibility is fully determined by the impact of trade on the activity cutoff. As the free-entry condition implies that the economy's total income and expenditure is given by total labor income, $E = \overline{w}L$, rearranging equation (21) yields $w^i(\underline{s})/P^i = \frac{(\sigma-1)}{\sigma} A(\underline{s}, \phi_i^*)[L/\sigma f]^{\frac{1}{\sigma-1}}$ for $i = a, \tau$, so trade rises the real wage of even the least-skilled workers in the economy if and only if it rises the activity cutoff. Note that this observation, together with proposition 4, implies that opening to international trade raises the real wage of the poorest workers in the economy only if it also induces a pervasive rise in wage inequality.

6.2 Trade Liberalization in the Free-Entry Model

The effects of a trade liberalization on the wage distribution in the free-entry-model can be derived by resorting to the results in propositions 2 to 4, as they largely cover the range of possible outcomes in this case. For the same reasons behind the corresponding result in proposition 4, a trade liberalization could lead to a rise or a fall in the activity cutoff. If the activity cutoff increases, then the situation is identical to that considered in proposition 3. If the activity cutoff declines, then the pre- and post-liberalization matching functions must intersect at least once on $(\underline{s}, \overline{s})$ to avoid a violation of the free entry condition as discussed in the case of proposition 4.ii. However, in the case of a trade liberalization, more than one crossing on $(\underline{s}, \overline{s})$ cannot be ruled out even when the conditions on the functions $\eta_0^F(t, \lambda)$ and $\eta_1^F(t, \lambda)$ in proposition 3 are satisfied.

[38] Formally, to the right of the first intersection point, the matching functions of the closed and open economies can be conceived as solutions to particular parameterizations of the general BVP (20) with $K_1 = 0$ that differ only in the parameter function $\alpha(\phi)$, which is constant in the former and increasing in the latter. The result then follows from a direct application of lemma 4.i in the appendix.

7 Empirically Relevant Distributional Effects of Trade

The analysis of the previous sections shows that an increase in trade openness generally has ambiguous theoretical implications for the wage distribution.[39] The goal of this section is to explore which of the theoretical possibilities described in that analysis are the most empirically relevant. To that end, I calibrate the primitives of the framework based on estimates from the literature and some broad features of firm data from Portugal. As much of the theoretical ambiguity is driven by the extensive-margin channel, a crucial target of the calibration is the fraction of firms that export in each decile of the empirical distribution of firms by value added per worker.

7.1 Data and Calibration

I calibrate the model's primitives based on estimates from the literature and moments in manufacturing firm data from Portugal for the year 2006. In particular, I compute all the empirical moments targeted in my calibration from a summary of the dataset constructed in **?**), which in turn draws from annual information on Portuguese firms reported under the *Informação Empresarial Simplificada*. For each decile of manufacturing firms in terms of value added per worker, this summary includes information on total employment, total labor costs, average wages, the share of firms that are exporters, and average value added per worker across firms. To match the choice of numeraire in the model, I normalize nominal values in the data by the average wage paid by firms. I briefly sketch my calibration approach below, leaving the details to appendix C.

My calibration approach is partly based on Melitz and Redding (2015), henceforth MR. Specifically, as in MR, I set the elasticity of substitution between final goods to four, $\sigma = 4$, and make the model match the average exports-to-sales ratio among Portuguese manufacturing firms, $n\tau^{1-\sigma}/(1 + n\tau^{1-\sigma}) = 0.31$, which yields a value for $n\tau^{1-\sigma}$.[40] As I explain in section C.6 of the appendix, all relevant calibrated variables—including the moments targeted in the calibration as well as wage inequality in the calibrated equilibrium—depend on $\{n, \tau\}$ only through $n\tau^{1-\sigma}$. The same is true regarding the counterfactual implications of the calibrated model discussed in the next section. As such, I proceed without picking specific values for $\{n, \tau\}$, as such a choice does not affect any of the results discussed below.[41]

Following MR, I make assumptions that guarantee that firms' revenue in the model, r^d, is distributed Pareto with shape parameter equal to 1. In particular, I assume that the CDF of firm productivity, $G(\phi)$, is that of a truncated Pareto distribution with shape parameter θ and that $r^d(\phi)$ is proportional to ϕ^θ in the calibrated equilibrium. In the appendix, I show that these assumptions impose restrictions on other model's elements, as the endogenous revenue function r^d depends on the productivity function $A(s, \phi)$ and on the shape of the equilibrium matching function, which in turn depends on other primitives, including the distributions of worker skill, firm productivity, and fixed export costs.[42] A convenient functional form

[39] The distributional effects are unambiguous only for the case of opening to trade in the no-free-entry model.

[40] In MR, $n = 1$, so this moment condition yields a value for τ.

[41] The reader shoud note, however, that the level variable trade costs determines the scope for further trade liberalization in the model, which could affect the interpretation of some counterfactual results.

[42] For example, the functional form $A(s, \phi) = B_0^A \exp\left(B_1^A s^{\alpha_s} \phi^{\alpha_\phi}\right)$ is not compatible with these restrictions.

for the productivity function that is compatible with these restrictions is $A(s,\phi) = B_0^A[\alpha_s s^\rho + \alpha_\phi \phi^\rho]^{B_1^A/\rho}$. According to this specification, $A(s,\phi)$ is homogeneous of degree $B_1^A > 0$, $\rho < 0$ is the constant elasticity of substitution between worker skill and firm productivity, and the positive parameters $\{B_0^A, \alpha_s, \alpha_\phi\}$ are overall and input-specific productivity shifters.[43]

To facilitate the estimation of the model, I also assume a functional form for the endogenous fraction of exporters as a function of firm productivity in the calibrated equilibrium, $FX(\phi) \equiv F(r^d(\phi)\tau^{1-\sigma}/\sigma f^x)$. Specifically, I assume that $FX(\phi)$ is given by the CDF of a truncated Pareto distribution with shape parameter γ and support in the interval $[\phi_{lb}^{FX}, \phi_{ub}^{FX}]$. The restrictions imposed by the assumptions on $\{G(\phi), r^d(\phi), A(s,\phi), FX(\phi)\}$ and the model's equilibrium conditions allow me to back out the implied functional forms of all remaining endogenous and exogenous elements of the model, including those of the exogenous distributions of worker skill and fixed export costs. As a result, I can compute the model's implications for several moments of the Portuguese data described above.

In the calibration of the model, I target the distribution of (i) total employment and (ii) the total wage bill across deciles of value added per worker, as well as (iii) the fraction of firms that export and (iv) the average value added per worker in each decile. As I show in sections C.5 and C.6 of the appendix, the model-implied values for these moments depend only on the parameters of the productivity distribution among active firms, $\{\phi^*, \overline{\phi}, \theta\}$, on the parameter B_1^A of the productivity function, and on the parameters of the assumed functional form for $FX(\phi)$, $\{\gamma,\ \phi_{lb}^{FX},\ \phi_{ub}^{FX}\}$. Moreover, these parameters completely determine the wage distribution in the calibrated equilibrium. Accordingly, in the calibration exercise, all remaining parameters are normalized or chosen to satisfy equilibrium conditions of the model. Noting that the selection of $\{\phi^*, \overline{\phi}\}$ is equivalent to a choice of measurement units for firm productivity, I also normalize the values of these parameters. Accordingly, I estimate $\{\theta,\ B_1^A,\ \gamma,\ \phi_{lb}^{FX},\ \phi_{ub}^{FX}\}$ via simulated methods of moments (SMM), targeting moments (i)-(iv) in the Portuguese data. Armed with all these parameters values, I pick the mass of firms in the no-free-entry model, $\overline{M}$, and the fixed entry costs in the free-entry model, f^e, to guarantee that the models are consistent with the normalized value for the activity cutoff ϕ^*.[44]

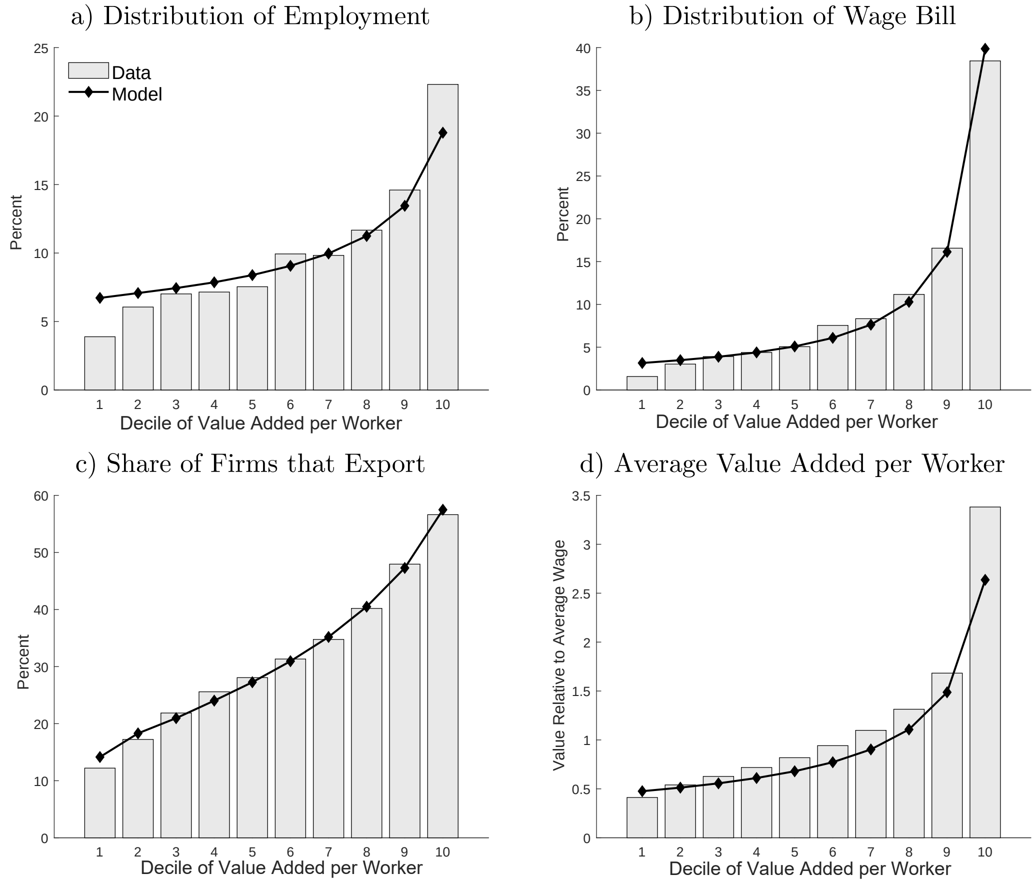
Despite being highly stylized, the model does a good job at fitting the targeted moments (i)-(iv) in the Portuguese data as shown in figure 4. In particular, the calibrated model fits particularly well the fraction of firms that export in each decile of the distribution of firms' value added per worker (panel c), a crucial target of the calibration. This moment plays a major role in pinning down the CDF of the firm-specific component of fixed export costs, the primitive of the model controlling the extensive-margin channel. As this margin drives much of the theoretical ambiguity regarding the distributional effects of higher trade openness, it is especially important that the calibrated model fits well this moment of the data.

The model also fits relatively well untargeted moments in the Portuguese data, including the average wage paid by firms in each decile of value added per worker and the distribution of total value added across these deciles (figure 6 in the appendix). In addition, the model implications for several measures

[43] The assumption $\rho < 0$ guarantees that $A(s,\phi)$ is strictly log-supermodular.

[44] These parameters are recovered from equilibrium conditions (19) and (22), respectively.

Figure 4: Model vs. Data



Note: For the moments targeted in the calibration of the model, the figure shows the model's prediction (line) and the target values computed from the Portuguese data for 2006 described in the text (bars).

of wage inequality not targeted in the calibration are in the ballpark of the values reported in Pereira (2021) as indicated in table 1 of the appendix.

Despite not affecting wage inequality in the calibrated equilibrium, the elasticity of substitution between worker skill and firm productivity in the productivity function $A(s, \phi)$, ρ, does affect the distributional effects of changes in trade costs. In particular, when s and ϕ are hard to substitute (lower values of ρ), a given change in trade costs is associated with less labor reallocation across firms and larger changes in relative wages.[45] As a rigorous estimation of ρ is beyond the scope of this paper, I explore the implications of the model for different values of ρ. The baseline results discussed in this section correspond to $\rho = -10$. For this value of ρ, the largest liberalization I consider—which is significantly larger than liberalizations typically featured in the literature—induces a change in the Gini index of about ten points, which is somewhat higher than the six-point range of variation of Portugal's Gini index over the last two

[45] In this case, larger changes in relative wages are required for firms to change their optimal choice of worker type.

decades.[46] In appendix D, I show that the main messages go through for $\rho = -5, -15$.[47].

7.2 Revisiting the Distributional Effects of Trade

Armed with calibrated parameter values, I revisit the distributional effects of a higher trade openness implied by the framework, focusing on those cases with ambiguous theoretical effects. Under the weak assumptions of the theoretical analysis in section 5, *a trade liberalization in the no-free-entry model* could lead to either pervasively higher wage inequality (matching functions do not cross) or higher inequality among the poorest workers combined with lower inequality elsewhere in the wage distribution (matching functions cross), with the ambiguity largely driven by extensive margin channel. In the calibrated model, only the first case is possible. Indeed, it is easy to check that the calibrated CDF of fixed exports costs, $F(y)$, satisfies the sufficient condition in proposition 3.iii.[48] As such, *any* decline in variable trade costs raises wage inequality pervasively in the calibrated no-free-entry model, regardless of its magnitude, initial level of trade costs or the parameter values of the productivity function $A(s, \phi)$ (including ρ).

To gain further insight on the implications of the calibrated model, I quantify the effects of trade-costs declines on overall wage inequality through each of the three channels defined in section 5—selection-into-activity, intensive-margin, and extensive-margin channels. Panel (a) of figure 5 shows the *incremental* change in the Gini index (black dots) and the contribution of each of these channels (stacked bars) as variable trade costs are incrementally reduced by the same proportion $\widehat{\tau}_{step} \equiv \frac{\tau_{post}}{\tau_{pre}}$, where τ_{pre} and τ_{post} are, respectively, the level of trade costs prevailing before and after the liberalization.[49] For example, the height of the first black dot in the chart captures the change in the Gini index induced by a decline in trade costs from their value in the calibrated equilibrium, τ_0, to $\tau_1 = \widehat{\tau}_{step}\tau_0$. Similarly, the height of the second one indicates the additional change in the Gini index as trade costs further decline to $\tau_2 = \widehat{\tau}_{step}\tau_1$. The horizontal axis of the chart indicates the cumulative decline in trade costs after k sequential liberalizations, $\widehat{\tau} = [\widehat{\tau}_{step}]^k$. Panel (b) shows the cumulative change in the Gini index—i.e., the values in panel (b) are the cumulative sum of those in panel (a).

A few lessons follow from figure 5. First, as trade costs decline, the boost to the Gini index is not uniform, increasing initially but moderating after trade costs reach about 30 percent of their initial level, $\widehat{\tau} \approx 0.3$. The second lesson relates to the *relative* quantitative importance of each of the three channels in affecting inequality. Notably, the contribution of the extensive-margin channel (always negative in the figure) is dwarfed by the combined (positive) contributions of the of the other two channels. In addition, the selection-into-activity channel increasingly dominates as trade costs decline. Specifically, while the contribution of the intensive margin gradually declines until vanishing, that of the selection-into-activity

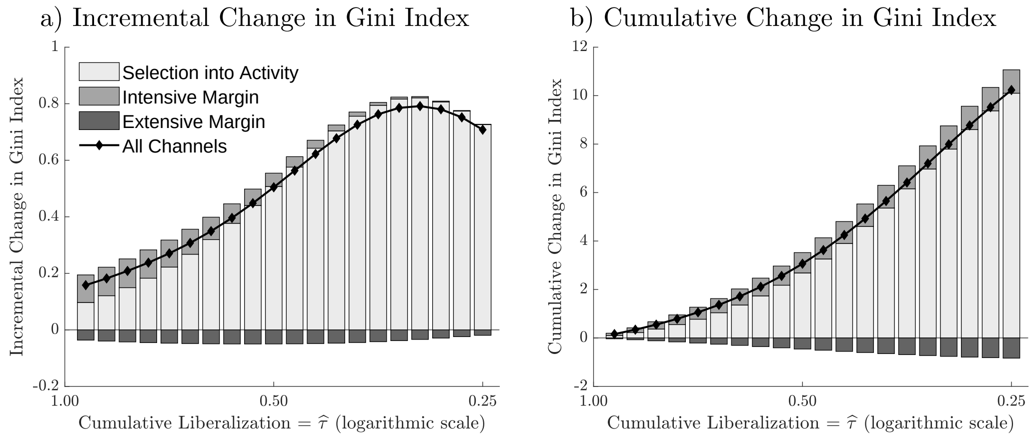
[46] World Bank estimates of Portugal's Gini index, which start in 2003, show a maximum value of 38.9 in 2004 at 38.9 and a minimum value of 32.8 in 2019.

[47] Although the effects of higher trade openness on wage inequality through each of the channels defined earlier are magnified for lower values of ρ, the *relative* quantitative importance of each of the channels is largely unchanged. As such, the conclusions about the most likely qualitative effects of trade on wage inequality are also unchanged.

[48] See section C.6 of the appendix for a derivation of this result.

[49] Specifically, trade costs decline by about 7 perent in each liberalization, $\widehat{\tau}_{step} \approx 0.93$, which follows from dividing the maximum cummulative decline in trade costs considred in the chart (75 percent) into 20 liberalization steps of the same size, $0.25 = (\widehat{\tau}_{step})^{20}$.

Figure 5: Trade Liberalization in the No-Free-Entry Model



Note: The figure illustrates the distributional effects of trade liberalizations in the calibrated no-free-entry model, decomposing total effects into the contributions of each of the three channels defined in section 5—selection-into-activity, intensive-margin and extensive-margin channels. Panel (a) shows the incremental change in the Gini index (black dots) and the contribution of each of these channels (stacked bars) as variable trade costs are incrementally reduced by same proportion $\widehat{\tau}_{step} \approx 0.93$. The horizontal axis indicates the cumulative decline in trade costs after k sequential liberalizations, $\widehat{\tau} = [\widehat{\tau}_{step}]^k$. Panel (b) shows the corresponding cumulative changes in the index and cumulative contributions of each channel.

channel increases for the most part, remaining significant in the range of cumulative trade costs declined considered in the figure. The general picture painted by figure 5 for the case of the Gini index also holds for other measures of wage inequality, as indicated by figure 7 of the appendix for the cases of the 90/10, 90/50 and 50/10 ratios.

As discussed in section 6, assuming *free entry* brings an additional source of theoretical ambiguity relative to the no-free-entry case, as the effects of increased trade openness on the activity cutoff cannot be determined without imposing additional restrictions on primitives. That said, for the calibration of the free-entry model discussed above, a decline in trade costs always induces a rise in the activity cutoff, so the distributional effects are qualitatively the same as those described earlier for the calibrated no-free-entry model. That is, inequality increases pervasively after a decline in trade costs, with the general messages from figure 5 also applying to this case.

The results of this section suggest that a decline in trade costs is likely to lead to pervasively higher wage inequality, in both the short and long run, through the labor-reallocation mechanisms emphasized in this paper. That said, the calibration exercise also shows that *a decline in trade costs always raises the real wage all workers.*

7.3 The importance of Accurately Quantifying the Extensive-Margin Channel

In this section, I present a result that further stresses the importance of carefully quantifying the extensive margin-channel of the model when assessing the distributional effects of international trade. Specifically, I show that changing the specification of fixed export costs in the calibration above to one in which all firms face the same cost f_x^c, a standard assumption since Melitz (2003), significantly affects the predictions of the model. Of note, in this alternative specification of the model, I isolate impact of this change in fixed-export-costs assumptions by choosing the level of f_x^c and the exogenous mass of firms, $\overline{M}^c$, such that the share of all firms that export and the activity cutoff remain unchanged relative to the baseline calibration. All other primitives of the model are unchanged.

For this alternative specification of the model, figure 8 in the appendix decomposes the effects of trade-costs declines on several measures of wage inequality into the same channels considered in figures 5 and 7 for the baseline model. These figures reveal major differences in the contribution of the extensive-margin channel across these two specifications. Notably, with common fixed export costs across firms, this channel exerts a much stronger downward pressure on wage inequality for initial liberalizations (before all firms export). Indeed, in some cases, this channel is strong enough to induce a decline in inequality among more skilled workers (crossing of matching functions), leading to slight declines in the 90/50 ratio. In contrast, this channel has a much less significant role in the baseline specification of the model, so a decline in trade costs always leads to a pervasive rise in wage inequality.

8 Conclusion

This paper develops a framework for studying the effects of higher trade openness on the wage distribution in which strong skill-productivity complementarities in production imply that inequality rises as workers reallocate towards more productive (skill-intensive) firms in the same industry. The model features a large number of skill groups and can accommodate weaker and more empirically relevant restrictions on firm selection into exporting than standard heterogenous-firms models. The cross-sectional structure of the model captures several features of the data identified by the trade and labor literatures. More productive firms tend to be larger, have workforces of higher average ability and pay higher average wages, and there is an imperfect correlation between firm size, wages and export status.

I use the framework to study the theoretical effects of higher trade openness on the wage distribution, decomposing these effects into those associated with the selection-into-activity, intensive-margin, and extensive-margin channels of trade, and considering two alternative assumptions about firm entry into the industry, no free entry a-lá Chaney (2008) and free entry a-lá Melitz (2003). In the no-free-entry model, opening to trade always leads to pervasively higher wage inequality. By contrast, a trade liberalization necessarily increases inequality at the lower end of the wage distribution, but may reduce it elsewhere. In the free-entry model, opening to trade leads to pervasively higher inequality (wage polarization) if low-productivity firms exit (enter) the market. In the case of a trade liberalization, all the previous possibilities could arise without additional restrictions on primitives. The analysis shows that much of this theoretical ambiguity is driven by the extensive-margin channel. In a calibrated version of the framework, this channel has a small quantitative role, so any increase in trade openness always leads to pervasively higher wage inequality. The analysis highlights the importance of properly accounting for the role of new exporters (extensive margin) in shaping the aggregate relative demand for skills, which in the framework is controlled by the specification of fixed export costs.

Finally, this paper contribute methodologically to the analysis of assignment problems. In addition to presenting existence and uniqueness results for a general BVP that encompasses those in this paper and others in the literature, I derive general results about the dependence of the solution to this BVP on parameters. These results can be used to analyze comparative statics exercises beyond those considered in this paper.

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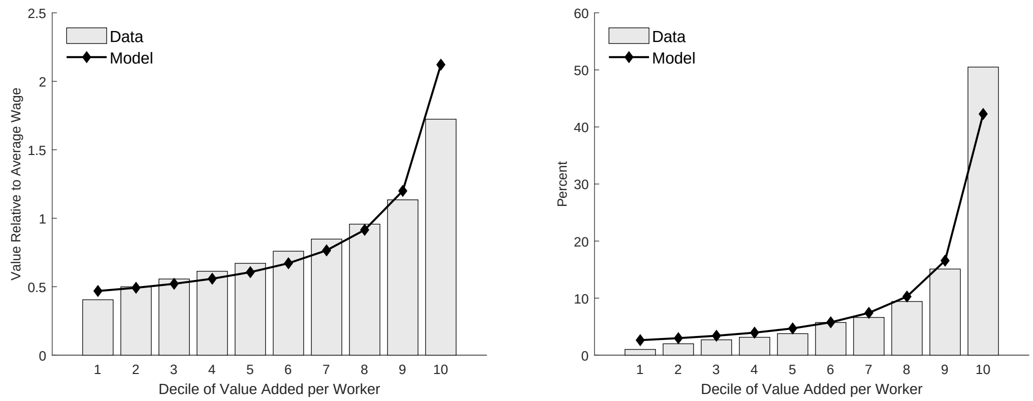
A Additional Figures and Tables

Figure 6: Model vs. Data, Nontargeted Moments

a) Average Wage b) Distribution of Total Value Added



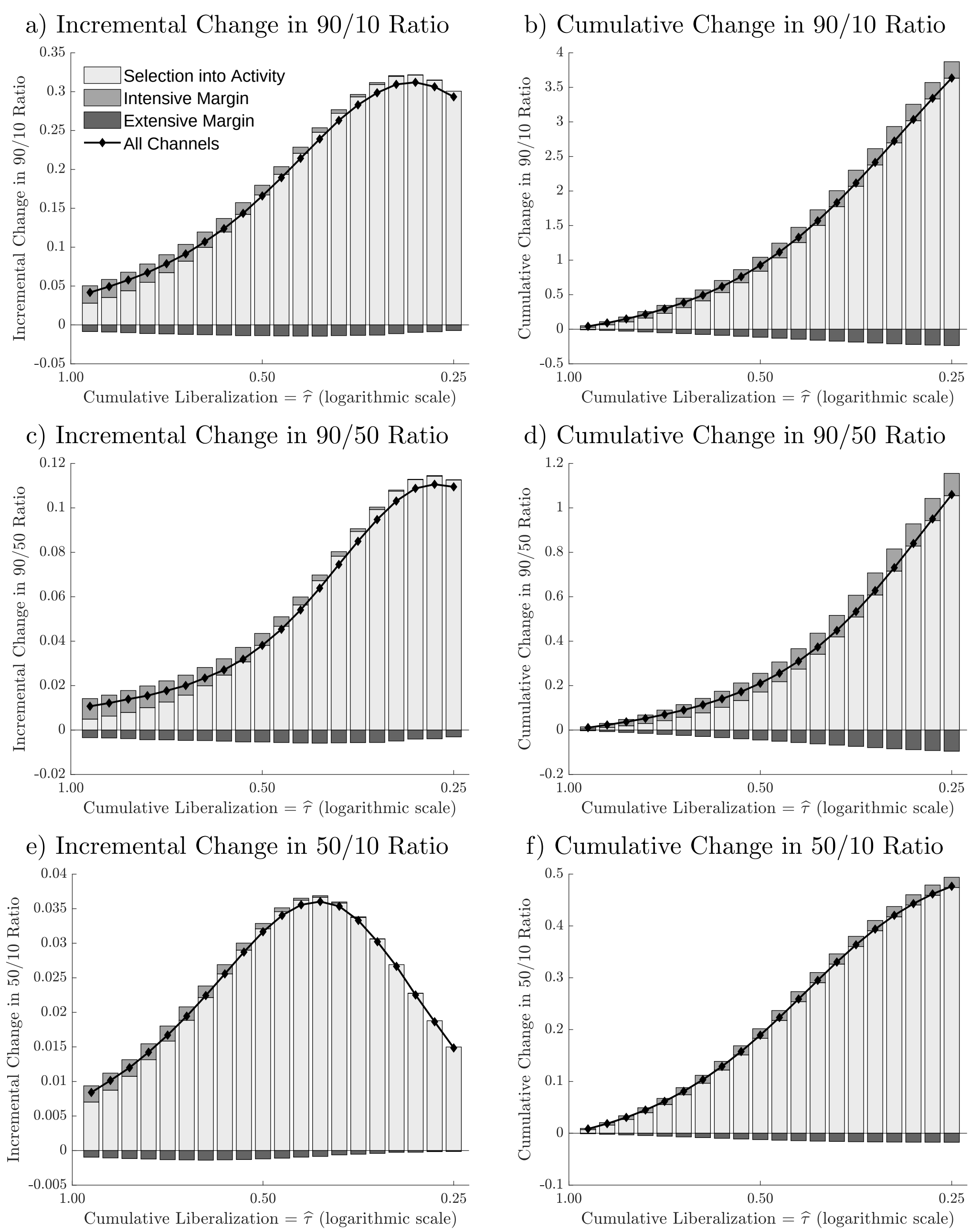
Note: For two moments not targeted in the calibration of the model, the figure compares the model's predictions (line) against their values in the Portuguese data for 2006 (bars) described in the text.

Table 1: Measures of Wage Inequality: Model vs. Data

	Model	Data	
		2005	2007
Gini Index	34	36	35
90/10 Ratio	4.75	4.06	3.97
90/50 Ratio	2.80	2.64	2.59
50/10 Ratio	1.69	1.53	1.53

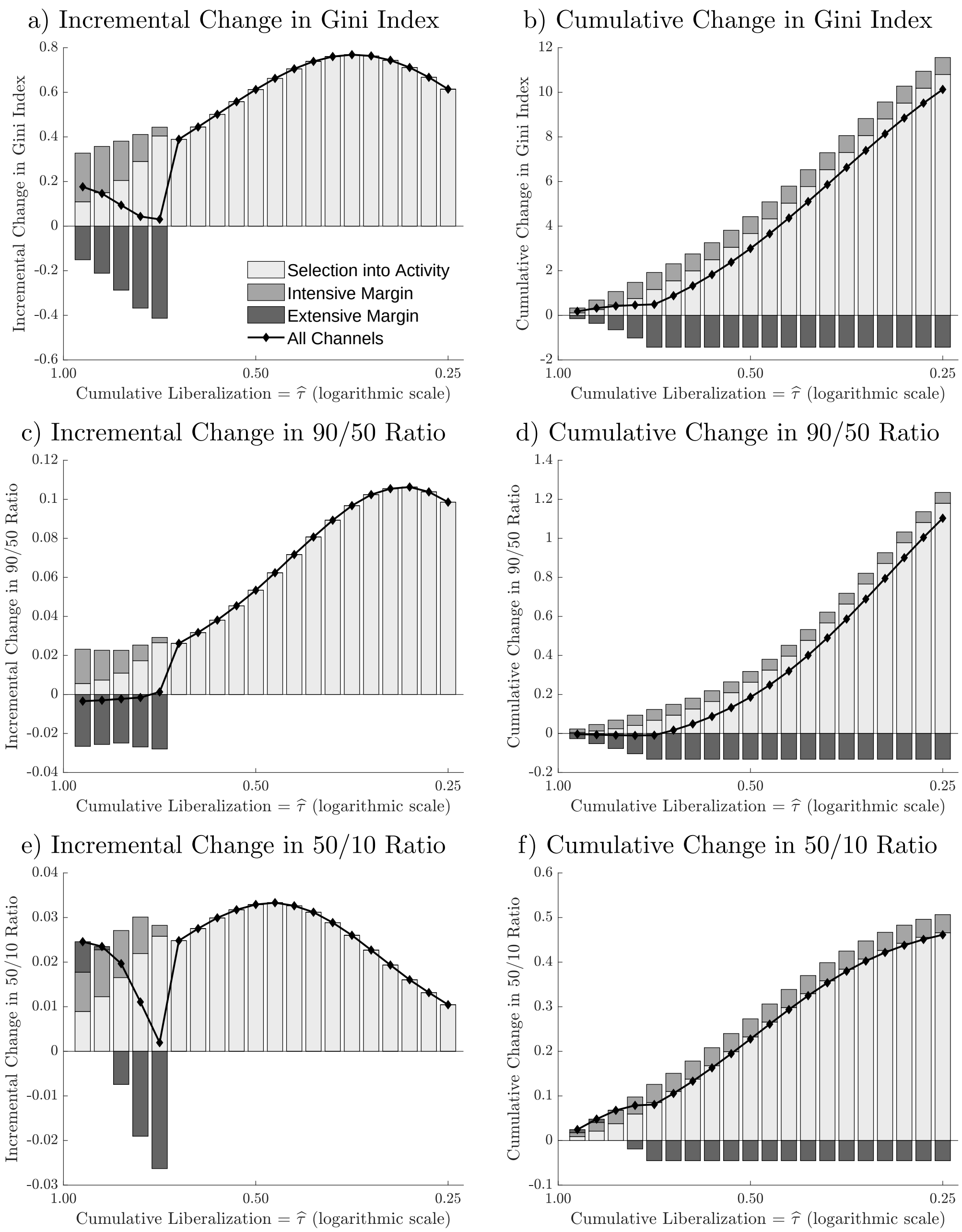
Note: The values in the first column correspond to the calibration of the model discussed in the text, which is based on manufacturing firm data from Portugal for 2006. Those in the second and third columns are taken from table 1 in Pereira (2021) and are based on wage data from the Portuguese dataset "Quadros de Pessoal" for the years 2005 and 2007. Values for 2006 are not reported in Pereira (2021). None of these inequality measure was targeted in the calibration of the model.

Figure 7: Trade Liberalization in the No-Free-Entry Model, Countinued



Note: The figure illustrates the distributional effects of trade liberalizations in the calibrated no-free-entry model, decomposing total effects into the contributions of each of the three channels defined in section 5—selection-into-activity, intensive-margin and extensive-margin channels. Panel (a) shows the incremental change in the 90/10 ratio (black dots) and the contribution of each of these channels (stacked bars) as variable trade costs are incrementally reduced by same proportion $\hat{\tau}_{step} \approx 0.93$. The horizontal axis indicates the cumulative decline in trade costs after k sequential liberalizations, $\hat{\tau} = [\hat{\tau}_{step}]^k$. Panel (b) shows the corresponding cumulative changes in the ratio and cumulative contributions of each channel. The rest of the panels show similar calculations for the 90/50 ratio (panels c and d) and the 50/10 ratio (panels e and f).

Figure 8: Trade Liberalization, Common Fixed Export Costs Across Firms



Note: The figure illustrates the distributional effects of trade liberalizations implied by the calibrated no-free-entry model under the alternative assumption of common fixed export costs across firms, decomposing total effects into the three channels defined in section 5 Panel (a) shows the incremental change in the Gini index (black dots) and the contribution of each of these channels (stacked bars) as variable trade costs are incrementally reduced by same proportion $\hat{\tau}_{step} \approx 0.93$. The horizontal axis indicates the cumulative decline in trade costs after k sequential liberalizations, $\hat{\tau} = [\hat{\tau}_{step}]^k$. Panel (b) shows the corresponding cumulative changes in the ratio and cumulative contributions of each channel. The rest of the panels show similar calculations for the 90/50 ratio (panels c and d) and the 50/10 ratio (panels e and f).

B Theoretical Appendix

B.1 Section 3

B.1.1 Proof of Lemma 1

Existence of a matching function N. I start by defining some notation. Let $S(\phi) \equiv \{s \in S : l(s,\phi) > 0\}$ and let $\Phi(s) = \left\{\phi \in \left[\phi^*, \overline{\phi}\right] : l(s,\phi) > 0\right\}$. To clarify the exposition of this part of the proof, I will proceed in a series of steps.

STEP 1: $\Phi(s) \neq \emptyset$ *for all* $s \in S$ *and* $S(\phi) \neq \emptyset$ *for all* $\phi \in \left[\phi^*, \overline{\phi}\right]$.

The full employment condition (9) and $V(s) > 0$ directly imply $\Phi(s) \neq \emptyset$ for all $s \in S$. Now suppose that we have an equilibrium in which there is $\phi \in \left[\phi^*, \overline{\phi}\right]$ such that $S(\phi) = \emptyset$. Then from (3) we have $q(\phi) = 0$ and this is incompatible with the demand given in (1), since for any $p(\phi) \in \mathbb{R}_+$ we have $q(\phi) > 0$. Then in any equilibrium we must have $S(\phi) \neq \emptyset$.

STEP 2: $S(.)$ *and* $\Phi(.)$ *satisfy the following properties: (i) if* $s \in S(\phi)$, $s' \in S(\phi')$ *and* $\phi' > \phi$, *then* $s' \geq s$*; and (ii) if* $\phi \in \Phi(s)$, $\phi' \in \Phi(s')$ *and* $s' > s$ *,then* $\phi' \geq \phi$.

(i) Suppose that this is not true and so let $s' < s$. Notice that (5) implies that $s \in S(\phi)$ if and only if $s \in \arg\min_z w(z)/A(z,\phi)$. Then $w(s)/A(s,\phi) \leq w(s')/A(s',\phi)$. In a similar way, $s' \in S(\phi')$ implies $w(s')/A(s',\phi') \leq w(s)/A(s,\phi')$. Combining both inequalities we get $A(s,\phi')A(s',\phi) \leq A(s,\phi)A(s',\phi')$, but this contradicts the log-supermodularity of A (remember that $\phi' > \phi$ and $s > s'$). Then we must have $s' \geq s$.

(ii) Suppose that this is not true and so let $\phi' < \phi$. Then $\phi \in \Phi(s) \Rightarrow s \in S(\phi)$ and $\phi' \in \Phi(s') \Rightarrow s' \in S(\phi')$. Then we have $\phi' < \phi$, $s \in S(\phi)$, $s' \in S(\phi')$ and by STEP 2.i this implies $s \geq s'$, which is a contradiction. Then we must have $\phi' \geq \phi$.

STEP 3: (i) $S(\phi)$ *is an interval for all* $\left[\phi^*, \overline{\phi}\right]$ *and* $|S(\phi) \cap S(\phi')| \leq 1$ *for any two different* $\phi, \phi' \in \left[\phi^*, \overline{\phi}\right]$*; (ii)* $\Phi(s)$ *is an interval for all* $s \in S$ *and* $|\Phi(s) \cap \Phi(s')| \leq 1$ *for any two different* $s, s' \in S$.

(i) I will prove the first part by contradiction. Suppose there is $\phi \in \left[\phi^*, \overline{\phi}\right]$ such that $S(\phi)$ is not an interval. Then there we can find $s, s' \in S(\phi)$, with $s < s'$, and some $s'' \in (s, s')$ such that $s'' \notin S(\phi)$. From STEP 1 we know that $\Phi(s'')$ is nonempty and so there must be a $\phi'' \in \left[\phi^*, \overline{\phi}\right]$ such that $s'' \in S(\phi'')$. We have only two possibilities: $\phi'' > \phi$ and $\phi'' < \phi$. If $\phi'' > \phi$, then STEP 2.i implies $s'' \geq s'$ which is a contradiction. If $\phi'' < \phi$, then STEP 2.i implies $s \geq s''$ which is also a contradiction. Then $S(\phi)$ is an interval for all $\left[\phi^*, \overline{\phi}\right]$.

Let us now show that $S(\phi) \cap S(\phi')$ is at most a singleton and as before I will proceed by contradiction. Suppose that the claim is not true. Then there must be $\phi, \phi' \in \left[\phi^*, \overline{\phi}\right]$ such that $s, s' \in S(\phi) \cap S(\phi')$ with $s \neq s'$. Without loss of generality assume $\phi' > \phi$ and $s' > s$. Then we have $\phi' > \phi$, $s' \in S(\phi)$, $s \in S(\phi')$ and so STEP 2.i implies $s \geq s'$ which is a contradiction. This concludes part i.

(ii) I prove this by contradiction. Suppose there is $s \in S$ such that $\Phi(s)$ is not an interval. Then there we can find $\phi, \phi' \in \Phi(s)$, with $\phi < \phi'$, and some $\phi'' \in (\phi, \phi')$ such that $\phi'' \notin \Phi(s)$. From STEP 1 we know that $S(\phi'')$ is nonempty and so there must be a $s'' \in S$ such that $\phi'' \in \Phi(s'')$. We have only two possibilities: $s'' > s$ and $s'' < s$. If $s'' > s$, then STEP 2.ii implies $\phi'' \geq \phi'$ which is a contradiction.

If $s'' < s$, then STEP 2.ii implies $\phi \geq \phi''$ which is also a contradiction. Then $\Phi(s)$ is an interval for all $s \in S$.

Let us now show that $\Phi(s) \cap \Phi(s')$ is at most a singleton and as before I will proceed by contradiction. Suppose that the claim is not true. Then there must be $s, s' \in S$ such that $\phi, \phi' \in \Phi(s) \cap \Phi(s')$ with $\phi \neq \phi'$. Without loss of generality assume $\phi' > \phi$ and $s' > s$. Then we have $s' > s$, $\phi' \in \Phi(s)$, $\phi \in \Phi(s')$ and so STEP 2.ii implies $\phi \geq \phi'$ which is a contradiction. This concludes part ii.

STEP 4: $S(\phi)$ is a singleton for all but a countable subset of $[\phi^*, \overline{\phi}]$.

I show this by contradiction. Let $\Phi_0 = \{\phi \in [\phi^*, \overline{\phi}] : |S(\phi)| > 1\}$ and suppose Φ_0 is uncountable. Notice that STEP 3.i implies that $S(\phi)$ is a nondegenerate interval for all $\phi \in \Phi_0$. Then for each $\phi \in \Phi_0$ we can pick a rational skill $r(\phi) \in intS(\phi)$ and given that $|S(\phi) \cap S(\phi')| \leq 1$ for any two different ϕ, ϕ' we must have $r(\phi) \neq r(\phi')$ when $\phi \neq \phi'$. Then the function $r : \Phi_0 \to \mathbb{Q} \cap S$ defined before is injective and so it is a contradiction since Φ_0 is uncountable.

STEP 5: $\Phi(s)$ is a singleton for all but a countable subset of S.

This follows from the same arguments as in STEP 4.

STEP 6: $S(\phi)$ is a singleton for all $\phi \in [\phi^*, \overline{\phi}]$.

I proceed by contradiction. Suppose there is $\phi \in [\phi^*, \overline{\phi}]$ such that $S(\phi)$ is not a singleton. Then STEP 3.i implies that $S(\phi)$ is an interval. By STEP 5 $\Phi(s) = \{\phi\}$ for all but a countable subset of $S(\phi)$. Then

$$l(s, \phi) = V(s)\, \delta\left[1 - I_{S(\phi)}\right] \text{ for almost all } s \in S(\phi)$$

where δ is the Dirac delta function. But then $q(\phi) = \int_{s \in S(\phi)} A(s, \phi)\, l(s, \phi)\, ds = \infty$, and this is incompatible with an equilibrium (as defined above). In other words, if $S(\phi)$ is not a singleton, then we would have a positive mass of workers producing in a single type of productivity firms which are of mass zero, and this cannot happen in equilibrium.

STEP 7: $\Phi(s)$ is a singleton for all $s \in S$.

I proceed by contradiction. Suppose there is an $s \in S$ such that $\Phi(s)$ is not a singleton. Then STEP 3.ii implies that $\Phi(s)$ is an interval. By STEP 6 $S(\phi) = \{s\}$ for all $\phi \in \Phi(s)$. Now let $\Phi_0 \subseteq \Phi(s)$ be the set of productivity levels that are assign a strictly positive *conditional*[50] mass of s-skill workers. I will show that Φ_0 is at most countable. The total conditional mass of s-skill workers allocated to productivities in Φ_0 can be expressed as

$$\int_{\Phi_0} l(s, \phi)\, d\phi = \int_{\phi^*}^{\overline{\phi}} k(\phi)\, \delta[1 - I_{\Phi_0}] d\phi$$

where δ is the Dirac delta function and $k(\phi)$ is the conditional mass of worker at productivity $\phi \in \Phi_0$. Notice that $\Phi_0 = \cup_{n=1}^{\infty} \{\phi \in \Phi_0 : k(\phi) \geq 1/n\}$ and because of the full employment condition each

[50] Remeber that the mass of workers of a particular skill s is zero. However, conditional on the skill, we can think of $l(s, \phi)$ as the density that represents the distribution of workers with skill s among the firms indexed by the productivity level. Then conditional on skill s, all s-skill workers have a total mass $V(s) > 0$. Then I say that a set $A \subseteq [\phi^*, \overline{\phi}]$ has possitive *conditional* mass if

$$\int_{\phi \in A} l(s, \phi)\, d\phi > 0.$$

$\{\phi \in \Phi_0 : k(\phi) \geq 1/n\}$ must be finite. Then Φ_0 is at most countable. This means a zero conditional mass of s-skill workers are allocated to almost all $\phi \in \Phi(s)$, which in turn means that $q(\phi) = 0$ for almost all $\phi \in \Phi(s)$. However this is incompatible with equilibrium since for any $p(\phi) \in \mathbb{R}_+$, the demand of variety ϕ (according to (1)) is strictly positive.

Steps 1,6,7 imply that there is a bijection $N : S \to [\phi^*, \overline{\phi}]$ such that $l(s, \phi) > 0$ if and only if $\phi = N(s)$ and by STEP 2 it must be strictly increasing.

Conditions i-iii. Consider a no-free-entry equilibrium of the closed economy with activity cutoff ϕ^*, wage schedule $w(s)$, price function $p(\phi)$, domestic revenue function $r^d(\phi)$ and matching function $N(s)$. The cost minimization condition (4) and the existence of the matching function N imply that $s = \arg\min_z w(z)/A(z, N(s))$, so $\frac{w(s)}{A(s,N(s))} \leq \frac{w(s+ds)}{A(s+ds,N(s))}$ and $\frac{w(s+ds)}{A(s+ds,N(s+ds))} \leq \frac{w(s)}{A(s,N(s+ds))}$. Combining these inequalities yields

$$\frac{A(s+ds, N(s))}{A(s, N(s))} \leq \frac{w(s+ds)}{w(s)} \leq \frac{A(s+ds, N(s+ds))}{A(s, N(s+ds))},$$

from which we can obtain the differentiability of $w(s)$ and equation (10), after taking logs, dividing by ds and taking limits as $ds \to 0$.[51] This proves **condition i**.

The pricing rule (5) and the existence of H imply $\phi = \arg\max_\gamma p(\gamma) A(H(\phi), \gamma)$, so

$$\begin{aligned} p(\phi) A(H(\phi), \phi) &\geq p(\phi + d\phi) A(H(\phi), \phi + d\phi), \\ p(\phi + d\phi) A(H(\phi + d\phi), \phi + d\phi) &\geq p(\phi) A(H(\phi + d\phi), \phi), \end{aligned}$$

Combining both inequalities yields

$$\frac{A(H(\phi), \phi + d\phi)}{A(H(\phi), \phi)} \leq \frac{p(\phi)}{p(\phi + d\phi)} \leq \frac{A(H(\phi + d\phi), \phi + d\phi)}{A(H(\phi + d\phi), \phi)}.$$

The differentiability of $p(\phi)$ and condition (11) are obtained taking logs, dividing by ds and taking limits as $ds \to 0$ in the last expression. Having established the differentiability of $p(\phi)$, the differentiability of $r^d(\phi)$ and condition (12) follow from the definition of $r^d(\phi)$ in (6).

The pricing rule (5) implies that the variable production cost of a firm equals a fraction $(\sigma - 1)/\sigma$ of its revenue. Then, the total wages paid to production workers employed at firms with productivity weakly lower than ϕ must be equal to a fraction $(\sigma - 1)/\sigma$ of the total revenue generated by those firms,

$$\int_{\underline{s}}^{H(\phi)} w(s) V(s) [L - fM] \, ds = \tfrac{(\sigma-1)}{\sigma} \int_{\underline{\phi}}^{\phi} r^d(\phi') g(\phi') \, d\phi' \overline{M} \text{ for all } \phi \in [\phi^*, \overline{\phi}]. \tag{23}$$

Due to the continuity of the revenue function $r^d(\phi)$, the right hand side of (23) is a differentiable function of the limit of integration ϕ. Then, the left hand side must also be a differentiable function of ϕ, which together with the continuity of $V(s)$ and $w(s)$, implies that $H(\phi)$ is differentiable. Differentiating (23)

[51] The limits are well defined since all the functions involved are continuous.

with respect to ϕ and using the pricing rule (5) to substitute for the wage $w(s)$ yields condition (13). Concluding the proof of **condition ii**, the boundary conditions on H follow from the definition of the matching function, while the initial condition on $r^d(\phi)$ is just the the zero-profit condition for firms firms with productivity ϕ^*. Finally, **condition iii** follows from equation (23), evaluated at $\phi = \overline{\phi}$, and the numeraire assumption, $\int_{\underline{s}}^{\overline{s}} w(s) V(s) ds = 1$.

Let us turn to the **sufficient conditions for an equilibrium** stated in the last part of the lemma. Suppose that $\{\phi^*, p, r^d, H\}$ satisfy conditions (ii)-(iii) and define $N \equiv H^{-1}$, $M \equiv [1 - G(\phi^*)]\overline{M}$, $w(s) \equiv \frac{\sigma-1}{\sigma} A(s, N(s)) p(N(s))$, $q(\phi) \equiv \frac{r(\phi)}{p(\phi)}$, and $l(s,\phi) \equiv V(s)[L - fM]\delta(\phi - N(s))$, where $\delta(x)$ is the Dirac-delta function. In what follows I show that $\{M, \phi^*, w, p, q, l\}$ is a no-free-entry equilibrium of the closed economy.

The definitions of $w(s)$, M, and $l(s,\phi)$ above immediately imply that the pricing rule (5), condition (8) and the labor market clearing condition (9) are satisfied. The definition of $q(\phi)$ and equation (13) yield an expression for $q(\phi)$ in terms of H and primitives of the model. The same expression is obtained computing the right hand side of (3) using the labor allocation $l(s,\phi)$ constructed here, so condition (3) is satisfied. The initial condition on the function $r^d(\phi)$ in point ii of the lemma implies that the zero-profit condition (7) holds. Using the definition of w above to substitute for p in equation (13), we arrive at (23) after rearranging and integrating on both sides. Evaluating (23) at $\phi = \overline{\phi}$ and using condition iii of the lemma yields $\int_{\underline{s}}^{\overline{s}} w(s) V(s) ds = 1$, so the numeraire condition holds. Finally, the construction of $q(\phi)$ implies that the consumer's budget constraint is satisfied and, together with conditions (11) and (12), implies $q(\phi')/q(\phi) = [p(\phi')/p(\phi)]^{-\sigma}$, so conditions (1) and (2) hold. This concludes the proof of the lemma.

B.1.2 Matching function and Lorenz dominance

Consider two economies A and B with matching functions N^A and N^B such that $N^B(s) > N^A(s)$ for all $s \in [s_0, s_1] \subseteq [\underline{s}, \overline{s}]$. As discussed in the main text, the strict log-supermodularity implies $w^A(s')/w^A(s) < w^B(s')/w^B(s)$, for all $s' > s$ in $[s_0, s_1]$.

In this context, the poorest ρ fraction of workers in the interval $[s_0, s_1]$ is associated with a skill $s(\rho)$ given by

$$\rho = \int_{s_0}^{s(\rho)} V(s) ds \Big/ \int_{s_0}^{s_1} V(s) ds.$$

The Lorenz Curve is then

$$L(\rho) \equiv \int_{s_0}^{s(\rho)} w(s) V(s) ds \Big/ \int_{s_0}^{s_1} w(s) V(s) ds = \frac{\int_{s_0}^{s(\rho)} \frac{w(s)}{w(s(\rho))} V(s) ds}{\int_{s_0}^{s(\rho)} \frac{w(s)}{w(s(\rho))} V(s) ds + \int_{s(\rho)}^{s_1} \frac{w(s)}{w(s(\rho))} V(s) ds}$$

It is readily seen that this implies that $L^A(\rho) > L^B(\rho)$ for all $\rho \in (0,1)$. Finally, from Atkinson (1970) we know that Lorenz dominance is equivalent to *Normalized* Second-Order Stochastic Dominance.

B.2 Section 4

B.2.1 Definition of Equilibrium

Definition 2 *A no-free-entry equilibrium of the open economy is an activity cutoff ϕ^*, a mass of active firms $M > 0$, a mass of exporters $M^x(\phi) > 0$ for each productivity level $\phi \geq \phi^*$, output functions $q^d, q^x : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, labor allocations functions $l^d, l^x : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a price function $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and a wage schedule $w : S \to \mathbb{R}_+$ such that the following conditions hold,*
(i) consumers behave optimally, equations (1) and (2);
(ii) firms behave optimally given their technology, equations (3), (5), (7), (8) and (16);
(iii) goods and labor markets clear, equations (6), (15) and (17);
(iv) the numeraire assumption holds, $\overline{w} = 1$.

B.2.2 Characterization of Equilibrium

Lemma 3 *In a no-free-entry equilibrium of the open economy with activity cutoff $\phi^* \in (\underline{\phi}, \overline{\phi})$ the following conditions hold.*
(i) There exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$, (with inverse function H) such that (i) $l^d(s, \phi) + l^x(s, \phi) > 0$ if and only if $N(s) = \phi$, (ii) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$.
(ii) The wage schedule w is continuously differentiable and satisfies (10)
(iii) The price, domestic revenue and matching functions, $\{p, r^d, N\}$, are continuously differentiable. Given ϕ^*, the triplet $\{p, r^d, H\}$ solves the BVP comprising the system of differential equations {(11), (12), (18)} and the boundary conditions $r^d(\phi^*) = \sigma f$, $H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$.
(iv) The activity cutoff ϕ^* and the revenue function r^d satisfy (19).
Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$, and functions $p, r^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $H : [\phi^*, \overline{\phi}] \to S$ satisfy the conditions (iii)-(iv), then they are, respectively, the activity cutoff, the price function, the domestic revenue function, and the inverse of the matching function of a no-free-entry equilibrium of the open economy.

Proof. Adapt arguments in the proof of lemma 1. ■

B.2.3 Proof of Lemma 2

Existence. My approach to prove the existence of a solution to the BVP (20) relies on fixed-point methods. The first step in such an approach is to recast the BVP under consideration as a fixed point problem of some functional operator. To that end, I define the functional Ψ, mapping the space of continuous functions into itself, as follows

$$\Psi(y)(\phi) \equiv s_0 + [s_1 - s_0] \frac{\int_{\phi_0}^{\phi} h(t, y(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} dt}\right) K_1\right] dt}{\int_{\phi_0}^{\phi_1} h(t, y(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(y(u),u)}{\partial \phi} dt}\right) K_1\right] dt}, \tag{24}$$

where

$$h(t, y(t)) \equiv \frac{A(s_0, \phi_0)}{A(y(t), t)} \frac{V(s_0)}{V(y(t))} \frac{g(t)}{g(\phi_0)} \frac{\alpha(t)}{\alpha(\phi_0)}. \tag{25}$$

The following lemma states that the problem of finding a solution to the BVP (20) is equivalent to the problem of finding a fixed point of the functional Ψ.

Claim 1 *A function Γ belongs to a triplet $\{z, x, \Gamma\}$ solving BVP (20) if and only if it is a fixed point of the functional $\Psi : \mathbf{C}[\phi_0, \phi_1] \to \mathbf{C}[\phi_0, \phi_1]$ defined in (24)-(25).*

Proof. Let us start with the "only if" part of the lemma. Let $\{z, x, \Gamma\}$ be a solution to the BVP (20). It can be shown that each of the functions in the solution triplet must be strictly positive, that x and Γ must be strictly increasing, and that z must be strictly decreasing. Then, equation (20c) implies that for any $t \in (\phi_0, \phi_1]$ we can write

$$\begin{aligned}
\Gamma_\phi(t) &= \Gamma_\phi(\phi_0)\, h(t, \Gamma(t)) \frac{x(t)\, z(\phi_0) [1 + F(K_0 x(t)) K_1]}{x(\phi_0)\, z(t) [1 + F(K_0 x(\phi_0)) K_1]} \\
&= \Gamma_\phi(\phi_0) \frac{h(t, \Gamma(t))}{[1 + F(K_0) K_1]} e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du} [1 + F(K_0 x(t)) K_1],
\end{aligned}$$

where the second line is obtained using equations (20a)-(20b) and $x(\phi_0) = 1$. Integrating $\Gamma_\phi(t)$ between ϕ_0 and ϕ yields

$$\Gamma(\phi) = \Gamma(\phi_0) + \Gamma_\phi(\phi_0) \int_{\phi_0}^{\phi} \frac{h(t, \Gamma(t))}{[1 + F(K_0) K_1]} e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du} [1 + F(K_0 x(t)) K_1]\, dt.$$

Evaluating the last expression at $\phi = \phi_1$, using the boundary conditions on Γ and solving for $\Gamma_\phi(\phi_0)$ we get

$$\Gamma_\phi(\phi_0) = \frac{[s_1 - s_0]}{\int_{\phi_0}^{\phi_1} \frac{h(t, \Gamma(t))}{[1 + F(K_0) K_1]} e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du} [1 + F(K_0 x(t)) K_1]\, dt}.$$

The last two expressions, $x(t) = e^{(\sigma - 1) \int_{\phi_0}^{t} \frac{\partial \ln A(H(t), t)}{\partial \phi} dt}$, and the definition of Ψ in (24) yield $\Gamma = \Psi(\Gamma)$— i.e., Γ is a fixed point of Ψ.

Let us turn to the "if" part of the lemma. Let Γ be a fixed point of Ψ. If we define $x(\phi) = e^{(\sigma-1) \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du}$ and $z(\phi) = \frac{[1 + F(K_0) K_1] \alpha(\phi_0) g(\phi_0) \overline{M}}{A(s_0, \phi_0) V(s_0) \Gamma_\phi(\phi_0)} e^{-\int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u), u)}{\partial \phi} du}$, then it is easy to verify that $\{z, x, \Gamma\}$ is a solution to BVP (20). ■

Having recasted the BVP (20) as the problem of finding a fixed point of the functional Ψ defined in (24)-(25), the next step is to establish certain properties of this functional that permit the application of some fixed point theorem in the literature. I do so in the next lemma, in which I state that Ψ is a compact self-map on some closed and convex subset of Banach space.

Claim 2 *Let K be the convex and closed subset of $\mathbf{C}[\phi_0, \phi_1]$ given by*

$$K \equiv \{y \in \mathbf{C}[\phi_0, \phi_1] : s_0 \le y(\phi) \le s_1 \text{ for all } \phi \in [\phi_0, \phi_1]\}, \tag{26}$$

and let Ψ be the functional defined in (24)-(25). If $\{V, g, \alpha\}$ are continuous and A is continuously differentiable, then Ψ is a compact self-map on K.

Proof. By definition of Ψ, $\Psi(y)(\phi)$ is a strictly increasing function with $\Psi(y)(\phi_0) = s_0$ and $\Psi(y)(\phi_1) = s_1$, so $\Psi(y) \in K$—i.e., Ψ is a self-map on K. To show that Ψ is compact we have to show that $\Psi(K)$ is relatively compact. Per the Arzela-Ascoli theorem, it enough to show that $\Psi(K)$ is bounded and equicontinuous.

Let us start by showing that $\Psi(K)$ is bounded. To simplify notation, let's define the following constants:

$$\begin{aligned}
\overline{h} &\equiv \max_{\phi,y\in[\phi_0,\phi_1]\times[s_0,s_1]} h(\phi,y); \qquad \underline{h} \equiv \min_{\phi,y\in[\phi_0,\phi_1]\times[s_0,s_1]} h(\phi,y); \\
\overline{r} &\equiv \max_{\phi,y\in[\phi_0,\phi_1]\times[s_0,s_1]} \tfrac{\partial \ln A(y,\phi)}{\partial \phi}; \qquad \underline{r} \equiv \min_{\phi,y\in[\phi_0,\phi_1]\times[s_0,s_1]} \tfrac{\partial \ln A(y,\phi)}{\partial \phi}
\end{aligned}$$

Since $\{A, V, g, \alpha\}$ are continuous and strictly positive on $\Phi \times S \supseteq [\phi_0, \phi_1] \times [s_0, s_1]$, then the constants $\overline{h}$ and $\underline{h}$ are well-defined and are bounded away from zero. Similarly, the assumptions on A imply that $\frac{\partial \ln A(y,\phi)}{\partial \phi}$ is strictly positive and continuous on $\Phi \times S$, so $\overline{r}$ and $\underline{r}$ are also well-defined and bounded away from zero. Then for any $y \in K$ we have

$$|\Psi(y)(\phi)| \leq s_0 + \frac{[s_1 - s_0]}{(\phi_1 - \phi_0)} \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1-\phi_0)} (1 + K_1)(\phi - \phi_0) \leq s_0 + [s_1 - s_0] \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1-\phi_0)} (1 + K_1).$$

The last result implies $\|\Psi(y)\|_\infty \leq s_0 + [s_1 - s_0] \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1-\phi_0)} (1 + K_1)$, and given that the selection of $y \in K$ was arbitrary, we conclude that $\Psi(K)$ is bounded.

Let us now show that $\Psi(K)$ is equicontinuous. For any $y \in K$ and $\phi' > \phi$ we have

$$\begin{aligned}
\left|\Psi(y)(\phi') - \Psi(y)(\phi)\right| &\leq [s_1 - s_0] \frac{\int_{\phi}^{\phi'} h(t, \Gamma(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} dt}\right) K_1\right] dt}{\int_{\phi_0}^{\phi_1} h(t, H(t))\, e^{\sigma \int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} du} \left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{t} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} dt}\right) K_1\right] dt} \\
&\leq \frac{[s_1 - s_0]}{(\phi_1 - \phi_0)} \frac{\overline{h}}{\underline{h}} e^{\sigma \overline{r}(\phi_1-\phi_0)} (1 + K_1) \left|\phi' - \phi\right|.
\end{aligned}$$

Given that the selection of $y \in K$ was arbitrary, the last inequality implies that $\Psi(K)$ is equicontinuous on $[\phi_0, \phi_1]$. ■

Per the last two claims, the existence of a solution to the BVP (20) can be obtained as a direct application of the Schauder fixed point theorem (SFPT).[52] A function Γ belongs to a triplet $\{z, x, \Gamma\}$ solving BVP (20) if and only if Γ is a fixed point of the functional Ψ defined in (24)-(25). In addition, this functional is a compact self-map on the closed and convex set K defined in (26), so the SFPT implies that Ψ has a fixed point on K. Then, this fixed point is part of a solution to the BVP (20). Finally, the

[52] For a statement of the SFPT see O'Regan (2013) or **?**).

continuity of $\{A, V, g, \alpha, F\}$ and (20c) implies that Γ is continuously differentiable.

Uniqueness. I start by proving an intermediate result that is used later. The continuous differentiability of $\{V, g, \alpha, F\}$ and the twice continuous differentiability of A imply that the right-hand side of equations (20a)-(20c) are locally Lipschitz continuous with respect to $\{z, x, \Gamma\}$, as the relevant partial derivatives are bounded on bounded sets. Then, the initial value problem (IVP) given by the differential equations (20a)-(20c) and the initial conditions $x(\phi_0) = 1$, $\Gamma(\phi_0) = s_0$, $z(\phi_0) = z_0$, has at most one solution.

Let us turn to the uniqueness of the solution to the BVP (20). I proceed by contradiction. Suppose that there are two different solutions $\{z', x', \Gamma'\}$ and $\{z, x, \Gamma\}$ to the BVP (20). Then, the uniqueness result in the previous paragraph implies that $z'(\phi_0) \neq z(\phi_0)$, which, together with equation (20c), implies $\Gamma'_\phi(\phi_0) \neq \Gamma_\phi(\phi_0)$. Without loss of generality suppose $\Gamma'_\phi(\phi_0) < \Gamma_\phi(\phi_0)$, that is, $\Gamma(\phi) > \Gamma'(\phi)$ in some neighborhood (ϕ_0, c), with $c > \phi_0$. By assumption, we know that the functions Γ' and Γ must intersect at least once again on $(\phi_0, \phi_1]$, since $\Gamma(\phi_1) = \Gamma'(\phi_1)$. Let ϕ^+ be the first value to the right of ϕ_0 at which the functions Γ' and Γ intersect—i.e., $\phi^+ \equiv \inf\{\phi \in (\phi_0, \phi_1] : \Gamma'(\phi) = \Gamma(\phi)\}$, and notice that ϕ^+ is well-defined since Γ' and Γ are continuous. Given our assumptions, $\Gamma(\phi) > \Gamma'(\phi)$ for $\phi \in (\phi_0, \phi^+)$, which together with $\Gamma(\phi^+) = \Gamma'(\phi^+)$, implies that $\Gamma'_\phi(\phi^+) \geq \Gamma_\phi(\phi^+)$. This and $\Gamma'_\phi(\phi_0) < \Gamma_\phi(\phi_0)$ imply

$$\frac{\Gamma'_\phi(\phi^+)/\Gamma'_\phi(\phi_0)}{\Gamma_\phi(\phi^+)/\Gamma_\phi(\phi_0)} > 1. \tag{27}$$

As discussed above, Γ' and Γ are fixed points of the functional Ψ defined in (24), $Z(\phi) = \Psi(Z)(\phi)$, for $Z = \Gamma', \Gamma$, so $Z_\phi(\phi)$ can be obtained differentiating the right-hand side of (24). Doing so yields,

$$Z_\phi(\phi^+)/Z_\phi(\phi_0) = h(\phi^+, Z(\phi^+))\, e^{\sigma \int_{\phi_0}^{\phi^+} \frac{\partial \ln A(Z(u),u)}{\partial \phi} du} \frac{\left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{\phi^+} \frac{\partial \ln A(Z(u),u)}{\partial \phi} du}\right) K_1\right]}{\left[1 + F(K_0) K_1\right]},$$

for $Z = \Gamma', \Gamma$. Combining the last expression for both functions yields

$$\frac{\Gamma'_\phi(\phi^+)/\Gamma'_\phi(\phi_0)}{\Gamma_\phi(\phi^+)/\Gamma_\phi(\phi_0)} = e^{\sigma \int_{\phi_0}^{\phi^+} \left[\frac{\partial \ln A(\Gamma'(u),u)}{\partial \phi} - \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi}\right] du} \frac{\left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{\phi^+} \frac{\partial \ln A(\Gamma'(u),u)}{\partial \phi} du}\right) K_1\right]}{\left[1 + F\left(K_0 e^{(\sigma-1)\int_{\phi_0}^{\phi^+} \frac{\partial \ln A(\Gamma(u),u)}{\partial \phi} du}\right) K_1\right]} < 1, \tag{28}$$

where in the last expression I used the fact that $\Gamma'(\phi^+) = \Gamma(\phi^+)$, so $h(\phi^+, \Gamma'(\phi^+)) = h(\phi^+, \Gamma(\phi^+))$. The log-supermodularity of A, $\Gamma(\phi) > \Gamma'(\phi)$ for $\phi \in (\phi_0, \phi^+)$ and the fact that F strictly increasing imply that each of the terms on the right-hand side of the last expression is strictly less than 1. However, note that equation (28) contradicts equation (27), so it must be the case that there is only one solution to the BVP (20).

Condition i. Let $\{z^i, x^i, \Gamma^i\}$ be the unique solution to BVP (20) with $K_1 = 0$ and $s_0 = s_0^i$, for $i = a, b$ and $s_0^a > s_0^b$. To prove the result, I show that if Γ^a and Γ^b intersect at some point $\phi^+ \in (\phi_0, \phi_1)$, then there are functions y^i and w^i for $i = a, b$, such that $\{w^a, y^a, \Gamma^a\}$ and $\{w^b, y^b, \Gamma^b\}$ solve the same IVP on $[\phi_0, \phi_1]$ given by the system (20a)-(20c) and the same initial value at any $\phi \in (\phi_+, \phi_1)$. Then, the uniqueness result proved at the beginning of the previous section implies that $\{y^a, w^a, \Gamma^a\} = \{y^b, w^b, \Gamma^b\}$ on $[\phi_0, \phi_1]$, contradicting the initial initial assumption $s_0^a > s_0^b$.

Suppose that there is a $\phi^+ \in (\phi_0, \phi_1)$ and $\Gamma^a(\phi^+) = \Gamma^b(\phi^+) \equiv s^+$. If we define the functions $y^i, w^i : [\phi_0, \phi_1] \to \mathbb{R}_+$ as $y^i(\phi) = z^i(\phi)/x^i(\phi^+)$, $w^i = x^i(\phi)/x^i(\phi^+)$, it is readily seen that on $[\phi^+, \phi_1]$ and for $i = a, b$, $\{y^i, w^i, \Gamma^i\}$ is a solution to the BVP given by the system of differential equations (20a)-(20c) and boundary conditions $w(\phi^+) = 1$, $\Gamma(\phi^+) = s^+$, $\Gamma(\phi_1) = s_1$. As this BVP is just a particular case of BVP (20), it has a unique solution, implying that $\{y^a, w^a, \Gamma^a\} = \{y^b, w^b, \Gamma^b\}$ on $[\phi^+, \phi_1]$. Moreover, this result implies that $\{w^a, y^a, \Gamma^a\}$ and $\{w^b, y^b, \Gamma^b\}$ solve the same IVP on $[\phi_0, \phi_1]$ given by the system (20a)-(20c) and the same initial conditions at any $\phi \in (\phi_+, \phi_1)$, which is the desired result. The no-crossing result related to the inverses of Γ^i can be establish in a similar way.

Condition ii. Let $\phi_0^a > \phi_0^b$ and suppose that $x^a(\phi) \equiv x(\phi; \phi_0^a) \geq x(\phi; \phi_0^b) \equiv x^b(\phi)$ for some ϕ on $[\phi_0^a, \phi_1]$. From their definitions, it is clear that $x^a(\phi_0^a) < x^b(\phi_0^a)$, so let ϕ' be the first productivity level such that $x^a(\phi) = x^b(\phi)$. Notice that ϕ' is well defined due to the continuity of the functions involved and due to our initial assumption. By definition of ϕ', we have $x^a(\phi') = x^b(\phi')$ and $x^a(\phi) < x^b(\phi)$ for $\phi < \phi'$. This means that $x^a(\phi)$ is catching up to $x^b(\phi)$, so this and the log-supermodularity of A imply that there is a $\phi'' < \phi'$, such that $\Gamma^a(\phi) > \Gamma^b(\phi)$ on (ϕ'', ϕ'), that is, Γ^a and Γ^b must intersect at least once strictly to left of ϕ'. Let ϕ_- denote the productivity level corresponding to the first intersection of Γ^a and Γ^b that is **strictly** to the left of ϕ'. Notice that ϕ_- is well defined —due to the continuity of the functions Γ^i and the fact that $\Gamma^a(\phi_0^a) < \Gamma^b(\phi_0^a)$— and that $\phi_- < \phi'$. Similarly, let ϕ_+ denote the productivity level corresponding to the first intersection of Γ^a and Γ^b that is **weakly** to the right of ϕ'. Notice that ϕ_+ is also well defined and that $\phi_+ \geq \phi'$.

From the definitions above we have $\Gamma^a(\phi_-) = \Gamma^b(\phi_-)$, $\Gamma^a(\phi) > \Gamma^b(\phi)$ on (ϕ_-, ϕ_+) and $\Gamma^a(\phi_+) = \Gamma^b(\phi_+)$. Then $\Gamma_\phi^a(\phi_-) \geq \Gamma_\phi^b(\phi_-)$ and $\Gamma_\phi^a(\phi_+) \leq \Gamma_\phi^b(\phi_+)$, so

$$\frac{\Gamma_\phi^a(\phi_+)/\Gamma_\phi^a(\phi_-)}{\Gamma_\phi^b(\phi_+)/\Gamma_\phi^b(\phi_-)} \leq 1. \tag{29}$$

As discussed above, we can differentiate the right-hand side of (24) to get

$$\frac{\Gamma_\phi^i(\phi_+)}{\Gamma_\phi^i(\phi_-)} = h^i(\phi_-, \phi_+)\, e^{\sigma \int_{\phi_-}^{\phi_+} \frac{\partial \ln A\left(\Gamma^i(t), t\right)}{\partial \phi} dt} \frac{\left[1 + F\left(K_0 x^i(\phi_+)\right) K_1\right]}{\left[1 + F\left(K_0 x^i(\phi_-)\right) K_1\right]} \text{ for } i = a, b,$$

where h is defined in (25). This implies

$$\begin{aligned}\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} &= e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^a(t),t\right)}{\partial\phi}-\frac{\partial \ln A\left(\Gamma^b(t),t\right)}{\partial\phi}\right]dt}\frac{\left[1+F\left(K_0x^a\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^a\left(\phi_-\right)\right)K_1\right]}{\left[1+F\left(K_0x^b\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^b\left(\phi_-\right)\right)K_1\right]}\\ &> \frac{x^a\left(\phi_+\right)/x^a\left(\phi_-\right)}{x^b\left(\phi_+\right)/x^b\left(\phi_-\right)}\frac{\left[1+F\left(K_0x^a\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^a\left(\phi_-\right)\right)K_1\right]}{\left[1+F\left(K_0x^b\left(\phi_+\right)\right)K_1\right]/\left[1+F\left(K_0x^b\left(\phi_-\right)\right)K_1\right]},\end{aligned} \tag{30}$$

where the second line is obtained multiplying the right-hand side by $\exp\int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A\left(\Gamma^b(t),t\right)}{\partial\phi}-\frac{\partial \ln A\left(\Gamma^a(t),t\right)}{\partial\phi}\right]<1$. Per our definitions we have $x^a\left(\phi'\right)=x^b(\phi')$, $x^a\left(\phi_+\right)\geq x^b(\phi_+)$ ($\Gamma^a\left(\phi\right)\geq\Gamma^b\left(\phi\right)$ on $\left[\phi',\phi_+\right]$), and $x^a\left(\phi_-\right)<x^b(\phi_-)$, which together with (30), imply

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)}>1,$$

contradicting (29). Then, it must be the case that $x^a\left(\phi\right)<x^b\left(\phi\right)$ for all $\phi\in[\phi^a_0,\phi_1]$, which is the desired result.

B.2.4 Proof of Proposition 1

The proof of the existence and uniqueness of the equilibrium in the closed and open economies was laid out in the text. Here, I prove the (constrained) efficiency of the equilibrium, starting with the simpler closed-economy case.

Efficiency of the Equilibrium of the Closed Economy

Below I show that an allocation is an equilibrium of the closed economy if and only if it is a solution to the planner's problem

$$\begin{gathered}\max_{\phi^*,\widetilde{q}(\phi),\widetilde{H}(\phi)}\int_{\phi^*}^{\overline{\phi}}\widetilde{q}\left(\phi\right)^{\frac{\sigma-1}{\sigma}}g\left(\phi\right)\overline{M}d\phi\\ \text{subject to}\\ \int_{\phi^*}^{\phi}\frac{\widetilde{q}(\phi')}{A(\widetilde{H}(\phi'),\phi')}g\left(\phi'\right)d\phi'\overline{M}=\int_{\underline{s}}^{\widetilde{H}(\phi)}V(s)ds\left[L-f\left[1-G\left(\phi^*\right)\right]\overline{M}\right]\ \text{ for all }\phi\in\left[\phi^*,\overline{\phi}\right],\\ \widetilde{H}(\phi^*)=\underline{s};\ \widetilde{H}(\overline{\phi})=\overline{s},\end{gathered} \tag{31}$$

where the left- and right-hand sides of the integral equation represent, respectively, the total mass of workers required to produce $\widetilde{q}\left(\phi'\right)$ units of each variety with productivity below ϕ, and the total mass of workers employed in the production of said varieties. Differentiating both sides of the integral equation above with respect to ϕ yields the the following ordinary differential equation (ODE) for all $\phi\in\left[\phi^*,\overline{\phi}\right]$,

$$\widetilde{H}_\phi\left(\phi\right)=\frac{\widetilde{q}\left(\phi\right)g\left(\phi\right)\overline{M}}{A(\widetilde{H}\left(\phi\right),\phi)V(\widetilde{H}\left(\phi\right))\left[L-f\left[1-G\left(\phi^*\right)\right]\overline{M}\right]}\equiv h^H(\phi^*,\widetilde{q}\left(\phi\right),\widetilde{H}\left(\phi\right),\phi). \tag{32}$$

Moreover, if (32) is satisfied for all $\phi \in \left[\phi^*, \overline{\phi}\right]$, then we can recover the integral equation above by moving $V(\widetilde{H}(\phi))\left[L - f\left[1 - G(\phi^*)\right]\right]$ to the left-hand side before integrating both sides of the resulting expression between $\left[\phi^*, \phi'\right]$ for each ϕ'. That is, the integral equation is equivalent to the ODE in (32), with the latter being the version of the constraint I consider below.

Following chapter 9 of Luenberger (1969), if $\{\phi^*, \widetilde{q}, \widetilde{H}\}$ solves problem (31), then there is a function of bounded variation, λ^H, and a real number, μ^H, such that the Lagrangian,

$$L(\phi^*, \widetilde{q}, \widetilde{H}) = \int_{\phi^*}^{\overline{\phi}} \widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\, d\phi \overline{M} + \int_{\phi^*}^{\overline{\phi}} \left[\widetilde{H}(\phi) - \underline{s} - \int_{\phi^*}^{\phi} h^H(\phi^*, \widetilde{q}(\phi'), \widetilde{H}(\phi'), \phi') d\phi'\right] d\lambda^H(\phi) + \mu^H \left[\widetilde{H}(\overline{\phi}) - \overline{s}\right]$$

is stationary at $\{\phi^*, \widetilde{q}, \widetilde{H}\}$. Integrating by parts the term involving a double integral and using the fact that λ^H is differentiable, the Lagrangian can be expressed as[53]

$$L(\phi^*, \widetilde{q}, \widetilde{H}) = \begin{cases} \int_{\phi^*}^{\overline{\phi}} \widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\, d\phi \overline{M} + \int_{\phi^*}^{\overline{\phi}} \widetilde{H}(\phi)\, \lambda_\phi^H(\phi)\, d\phi + \lambda^H(\phi^*)\underline{s} - \widetilde{H}\left(\overline{\phi}\right) \lambda^H(\overline{\phi}) + \cdots \\ \cdots \int_{\phi^*}^{\overline{\phi}} h^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi) \lambda^H(\phi) d\phi + \mu^H \left[H\left(\overline{\phi}\right) - \overline{s}\right] \end{cases}$$

The stationarity condition, together with the constraints of the problem, yields the following first order necessary conditions for an optimum

$$\begin{gathered} \widetilde{H}_\phi(\phi) = h^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi) \\ h_H^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi) + \lambda_\phi^H(\phi) = 0 \\ \tfrac{\sigma-1}{\sigma} \widetilde{q}(\phi)^{-\frac{1}{\sigma}} g(\phi) \overline{M} + h_q^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi) = 0 \\ \left[\mu^H - \lambda^H\left(\overline{\phi}\right)\right] = 0 \\ \widetilde{H}(\phi^*) = \underline{s}, \quad \widetilde{H}(\overline{\phi}) = \overline{s} \\ \int_{\phi^*}^{\overline{\phi}} h_{\phi^*}^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi) d\phi = \widetilde{q}(\phi^*)^{\frac{\sigma-1}{\sigma}} g(\phi^*) \overline{M} + h^H(\phi^*, \widetilde{q}(\phi), \widetilde{H}(\phi), \phi)\lambda^H(\phi^*). \end{gathered} \tag{33}$$

The first five lines in (33) are the standard necessary conditions of optimal control theory and reflect the constraints of the problem and the implications of stationarity of the Lagrangian with respect to $\{\widetilde{H}, \widetilde{q}\}$. The last line in (33) follows from the stationarity with respect to ϕ^*. Below I show that if $\{\phi^*, \widetilde{q}, \widetilde{H}, \lambda^H\}$ satisfies (33), then we can define functions $\{\widetilde{p}(\phi), \widetilde{r}(\phi)\}$ such that $\{\phi^*, \widetilde{p}(\phi), \widetilde{r}(\phi), \widetilde{H}\}$ satisfy the conditions of lemma 1, proving that a solution to the planner's problem is an equilibrium of the closed economy.

Let $\{\phi^*, \widetilde{q}, \widetilde{H}, \lambda^H\}$ satisfy the conditions in (33). For some (still undefined) positive constant p_0, define

$$\widetilde{p}(\phi) \equiv p_0 \tfrac{\sigma}{\sigma-1} \frac{-\lambda^H(\phi)}{A(\widetilde{H}(\phi), \phi) V(\widetilde{H}(\phi))}, \tag{34}$$

[53] See section 9.5 of Luenberger (1969) for a derivation of the differentiability of λ^H.

which, together with (33), implies

$$\widetilde{p}_{\phi}(\phi) = -\widetilde{p}(\phi)\frac{\partial \ln A(\widetilde{H}(\phi),\phi)}{\partial\phi} \tag{35}$$

Using (34) in the third line of (33) yields

$\widetilde{q}(\phi) = p_0^{\sigma}\left[L - f\left[1 - G(\phi^*)\right]\overline{M}\right]^{\sigma}\widetilde{p}(\phi)^{-\sigma}$,

so defining

$$\widetilde{r}(\phi) \equiv \widetilde{q}(\phi)\,\widetilde{p}(\phi) \tag{36}$$

implies

$$\begin{aligned}\widetilde{r}(\phi) &= p_0^{\sigma}\left[L - f\left[1 - G(\phi^*)\right]\overline{M}\right]^{\sigma}\widetilde{p}(\phi)^{1-\sigma},\\ \widetilde{r}(\phi) &= p_0\left[L - f\left[1 - G(\phi^*)\right]\overline{M}\right]\widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}}\end{aligned} \tag{37}$$

and

$$\widetilde{r}_{\phi}(\phi) = (\sigma - 1)\,\widetilde{r}_{\phi}(\phi)\frac{\partial \ln A(\widetilde{H}(\phi),\phi)}{\partial\phi} \tag{38}$$

With these definitions, the the first line of (33) can be expressed as

$$\widetilde{H}_{\phi}(\phi) = \frac{\widetilde{r}(\phi)\,g(\phi)\,\overline{M}}{A(\widetilde{H}(\phi),\phi)V(\widetilde{H}(\phi))\widetilde{p}(\phi)\left[L - f\overline{M}\left[1 - G(\phi^*)\right]\right]}. \tag{39}$$

Finally, noting that the third line in (33) implies $\frac{\sigma-1}{\sigma}\widetilde{q}(\phi)^{\frac{\sigma-1}{\sigma}}g(\phi)\overline{M} = -\widetilde{H}(\phi)\lambda^H(\phi)$, the last line in (33) can be expressed as

$\int_{\phi^*}^{\overline{\phi}} -\widetilde{H}(\phi)\lambda^H(\phi)d\phi\frac{f\overline{M}g(\phi^*)}{\left[L-f[1-G(\phi^*)]\overline{M}\right]} = \widetilde{q}(\phi^*)^{\frac{\sigma-1}{\sigma}}g(\phi^*)\overline{M} + \widetilde{H}(\phi^*)\lambda^H(\phi^*)$,

$\int_{\phi^*}^{\overline{\phi}}\left[\frac{\widetilde{q}(\phi)}{\widetilde{q}(\phi^*)}\right]^{\frac{\sigma-1}{\sigma}}g(\phi)\,d\phi\sigma f\overline{M} = \frac{\sigma}{\sigma-1}\left[L - f\left[1 - G(\phi^*)\right]\overline{M}\right]$,

$$\sigma f\int_{\phi^*}^{\overline{\phi}}\frac{\widetilde{r}(\phi)}{\widetilde{r}(\phi^*)}g(\phi)\,d\phi\overline{M} = \frac{\sigma}{\sigma-1}\left[L - f\left[1 - G(\phi^*)\right]\overline{M}\right], \tag{40}$$

where the derivation uses (37). If we choose the constant p_0 in (34) such that $\widetilde{r}(\phi^*) = \sigma f$, then the last equation can be expressed as

$$\int_{\phi^*}^{\overline{\phi}}\widetilde{r}(\phi)\,g(\phi)\,d\phi\overline{M} = \frac{\sigma}{\sigma-1}\left[L - f\left[1 - G(\phi^*)\right]\right] \tag{41}$$

Note that conditions $\{(35),(38),(39),(41)\}$, $\{\widetilde{H}(\phi^*) = \underline{s}, \widetilde{H}(\overline{\phi}) = \overline{s}\}$, and $\widetilde{r}(\phi^*) = \sigma f$ are identical to those in lemma 1, so $\{\widetilde{p}, \widetilde{r}, \widetilde{H}\}$ are the price, revenue and inverse matching functions corresponding to the closed economy equilibrium.

On the other direction, let $\left\{\phi_a^*, p, r^d, H\right\}$ be the activity cutoff, price, revenue and inverse matching functions of the closed economy equilibrium, with output function $q^d(\phi) = r^d(\phi)/p(\phi)$. As $\left\{p, r^d, H\right\}$ satisfy the ODE (13), then $\left\{q^d, H\right\}$ satisfy the first condition in (33). Define $\lambda \equiv -\lambda_0\frac{\sigma-1}{\sigma}p(\phi)\,A(H(\phi),\phi)V(H(\phi))$

for some positive constant λ_0. Log-differentiating $-\lambda$, together with equilibrium condition (11), yields the second line in (33). Using these definitions in the third condition of (33) yields

$$q^d\left(\phi_a^*\right)^{\frac{\sigma-1}{\sigma}}\left[\frac{q^d\left(\phi\right)}{q^d\left(\phi_a^*\right)}\right]^{\frac{\sigma-1}{\sigma}}-\frac{r^d\left(\phi\right)}{\left[L-f\left[1-G\left(\phi_a^*\right)\right]\overline{M}\right]}\lambda_0=0.$$

Recalling that the CES demand system implies $r^d\left(\phi\right)=Bq^d\left(\phi\right)^{\frac{\sigma-1}{\sigma}}$ for some constant B, the last expression holds for

$$\lambda_0=\frac{\left[L-f\left[1-G\left(\phi_a^*\right)\right]\overline{M}\right]q^d\left(\phi^*\right)^{\frac{\sigma-1}{\sigma}}}{r^d\left(\phi_a^*\right)}.$$

Finally, the derivations above imply that $\left\{q^d,H,\lambda,\phi_a^*\right\}$ satisfy the last line in (33) if and only if they satisfy equation (40), a fact that follows from the zero profit condition $r^d\left(\phi_a^*\right)=\sigma f$ and the numeraire condition (14). Accordingly, $\left\{q^d,H,\lambda,\phi_a^*\right\}$ solves the planner's problem.

Efficiency of the Equilibrium of the Open Economy

In this section, I show that an allocation is an equilibrium of the open economy if and only if it is a solution to the planner's problem

$$\max_{\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}}\int_{\phi^*}^{\overline{\phi}}\widetilde{q}^d\left(\phi\right)^{\frac{\sigma-1}{\sigma}}g\left(\phi\right)\overline{M}d\phi+n\int_{\phi^*}^{\overline{\phi}}\widetilde{q}^x\left(\phi\right)^{\frac{\sigma-1}{\sigma}}F\left(\widetilde{y}\left(\phi\right)\right)g\left(\phi\right)\overline{M}d\phi$$

subject to

$$\begin{gathered}\int_{\phi^*}^{\phi}\frac{\widetilde{q}^d(\phi')}{A(\widetilde{H}(\phi'),\phi')}g\left(\phi'\right)d\phi'\overline{M}+nf_x\int_{\phi^*}^{\phi}\frac{\widetilde{q}^x(\phi')\tau}{A(\widetilde{H}(\phi'),\phi')}F\left(\widetilde{y}\left(\phi'\right)\right)g\left(\phi'\right)\overline{M}d\phi'=\cdots\\ \cdots\int_{\underline{s}}^{\widetilde{H}(\phi)}V(s)dsL^{pw}(\phi^*,\widetilde{y})\text{ for all }\phi\in\left[\phi^*,\overline{\phi}\right],\\ \widetilde{H}(\phi^*)=\underline{s};\;\widetilde{H}(\overline{\phi})=\overline{s}.\end{gathered}\tag{42}$$

where $L^{pw}(\phi^*,\widetilde{y})$ represents the mass of production workers,

$$L^{pw}(\phi^*,\widetilde{y})=\left[L-f\left[1-G\left(\phi^*\right)\right]\overline{M}-nf_x\int_{\phi^*}^{\overline{\phi}}\int_{y}^{\widetilde{y}(\phi')}ydF(y)g\left(\phi'\right)\overline{M}d\phi'\right]$$

As explained in the case of the planner's problem for the closed economy, the integral constraint is equivalent to the following ODE,

$$\begin{gathered}\widetilde{H}_\phi\left(\phi\right)=h^H(\phi^*,\widetilde{q}^d\left(\phi\right),\widetilde{q}^x\left(\phi\right),\widetilde{H}\left(\phi\right),\widetilde{y}\left(\phi\right),\phi),\\ h^H(\cdots,\phi)\equiv\frac{\left[\widetilde{q}^d\left(\phi\right)+\widetilde{q}^x\left(\phi\right)F\left(\widetilde{y}\left(\phi\right)\right)\tau n\right]g\left(\phi\right)\overline{M}}{A(\widetilde{H}\left(\phi\right),\phi)V(\widetilde{H}\left(\phi\right))L^{pw}(\phi^*,\widetilde{y})}.\end{gathered}\tag{43}$$

Following chapter 9 of Luenberger (1969), if $\{\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}\}$ solves problem (42), then there is a function of bounded variation, λ^H, and a real number, μ^H, such that the Lagrangian,

$$L(\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}) = \int_{\phi^*}^{\overline{\phi}} \widetilde{q}^d(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\overline{M}d\phi + n\int_{\phi^*}^{\overline{\phi}} \widetilde{q}^x(\phi)^{\frac{\sigma-1}{\sigma}} F(\widetilde{y}(\phi)) g(\phi)\overline{M}d\phi + \cdots$$
$$\cdots \int_{\phi^*}^{\overline{\phi}} \left[\widetilde{H}(\phi) - \underline{s} - \int_{\phi^*}^{\phi} h^H(\cdots,\phi')d\phi'\right] d\lambda^H(\phi) + \mu^H\left[\widetilde{H}\left(\overline{\phi}\right) - \overline{s}\right]$$

is stationary at $\{\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H}(\phi),\widetilde{y}\}$. Integrating by parts the term involving a double integral and using the fact that λ^H is differentiable, the Lagrangian can be expressed as

$$L(\phi^*,\widetilde{q}^d,\widetilde{q}^x,\widetilde{H},\widetilde{y}) = \begin{cases} \int_{\phi^*}^{\overline{\phi}} \widetilde{q}^d(\phi)^{\frac{\sigma-1}{\sigma}} g(\phi)\overline{M}d\phi + n\int_{\phi^*}^{\overline{\phi}} \widetilde{q}^x(\phi)^{\frac{\sigma-1}{\sigma}} F(\widetilde{y}(\phi)) g(\phi)\overline{M}d\phi + \cdots \\ \cdots \int_{\phi^*}^{\overline{\phi}} \widetilde{H}(\phi)\lambda_\phi^H(\phi)d\phi + \lambda^H(\phi^*)\underline{s} - \widetilde{H}\left(\overline{\phi}\right)\lambda^H(\overline{\phi}) + \cdots \\ \cdots \int_{\phi^*}^{\overline{\phi}} h^H(\cdots,\phi)\lambda^H(\phi)d\phi + \mu^H\left[H\left(\overline{\phi}\right) - \overline{s}\right] \end{cases}$$

The stationarity condition and the constraints of the problem yield the following first order necessary conditions for an optimum

$$\begin{gathered} \widetilde{H}_\phi(\phi) = h^H(\cdots,\phi) \\ h_H^H(\cdots,\phi)\lambda^H(\phi) + \lambda_\phi^H(\phi) = 0 \\ \tfrac{\sigma-1}{\sigma}\widetilde{q}^d(\phi)^{-\frac{1}{\sigma}} g(\phi)\overline{M} + h_{q^d}^H(\cdots,\phi)\lambda^H(\phi) = 0 \\ \tfrac{\sigma-1}{\sigma}\widetilde{q}^x(\phi)^{-\frac{1}{\sigma}} \tau n F(\widetilde{y}(\phi)) g(\phi)\overline{M} + h_{q^x}^H(\cdots,\phi)\lambda^H(\phi) = 0 \\ n\widetilde{q}^x(\phi)^{\frac{\sigma-1}{\sigma}} F_y(\widetilde{y}(\phi)) g(\phi)\overline{M}d\phi + \frac{F_y(\widetilde{y}(\phi))\tau^{1-\sigma}n}{[1+F(\widetilde{y}(\phi))\tau^{1-\sigma}n]} h^H(\cdots,\phi)\lambda^H(\phi) + \cdots \\ \cdots \tfrac{\int_{\phi^*}^{\overline{\phi}} h^H(\cdots,\phi)\lambda^H(\phi)d\phi}{L^{pw}(\phi^*,\widetilde{y})} n f_x \widetilde{y}(\phi) F_y(\widetilde{y}(\phi)) g(\phi)\overline{M} = 0 \\ \left[\mu^H - \lambda^H\left(\overline{\phi}\right)\right] = 0 \\ \widetilde{H}(\phi^*) = \underline{s}, \quad \widetilde{H}(\overline{\phi}) = \overline{s} \\ \int_{\phi^*}^{\overline{\phi}} h_{\phi^*}^H(\cdots,\phi)\lambda^H(\phi)d\phi = \left[\widetilde{q}^d(\phi^*)^{\frac{\sigma-1}{\sigma}} + n\widetilde{q}^x(\phi^*)^{\frac{\sigma-1}{\sigma}} F(\widetilde{y}(\phi^*))\right] g(\phi^*)\overline{M} + h^H(\cdots,\phi^*)\lambda^H(\phi^*). \end{gathered} \tag{44}$$

The first seven lines in (44) are the standard necessary conditions of optimal control theory and reflect the constraints of the problem and the implications of stationarity of the Lagrangian with respect to $\{\widetilde{H},\widetilde{q}^d,\widetilde{q}^x,\widetilde{y}\}$. The last line in (44) follows from the stationarity with respect to ϕ^*. Below, I show that if $\{\phi^*,\widetilde{H},\widetilde{q}^d,\widetilde{q}^x,\widetilde{y},\lambda^H\}$ satisfies (44), then we can define functions $\{\widetilde{p}(\phi),\widetilde{r}(\phi)\}$ such that $\{\phi^*,\widetilde{p}(\phi),\widetilde{r}(\phi),\widetilde{H}\}$ satisfy the conditions of lemma 3 in the appendix, proving that a solution to the planner's problem is an equilibrium of the open economy.

Let $\{\phi^*,\widetilde{H},\widetilde{q}^d,\widetilde{q}^x,\widetilde{y},\lambda^H\}$ satisfy the conditions in (44). The third and fourth lines in (44) yield $\widetilde{q}^x(\phi) = \widetilde{q}^d(\phi)\tau^{-\sigma}$. For some (still undefined) positive constant p_0, define

$$\widetilde{p}(\phi) \equiv p_0 \tfrac{\sigma}{\sigma-1} \frac{-\lambda^H(\phi)}{A(\widetilde{H}(\phi),\phi)V(\widetilde{H}(\phi))}, \tag{45}$$

which, together with the second line in (44), implies

$$\widetilde{p}_{\phi}(\phi) = -\widetilde{p}(\phi)\frac{\partial \ln A(\widetilde{H}(\phi),\phi)}{\partial \phi} \tag{46}$$

Using (45) in the third condition in (44) yields

$\widetilde{q}(\phi) = p_0^{\sigma} L^{pw}(\phi^*, \widetilde{y})^{\sigma} \widetilde{p}(\phi)^{-\sigma}$.

Accordingly, defining

$$\widetilde{r}^d(\phi) \equiv \widetilde{q}^d(\phi)\,\widetilde{p}(\phi) \tag{47}$$

we get

$$\begin{aligned} \widetilde{r}^d(\phi) &= p_0^{\sigma}\left(L^{pw}(\phi^*, \widetilde{y})\right)^{\sigma} \widetilde{p}(\phi)^{1-\sigma}, \\ \widetilde{r}^d(\phi) &= p_0 L^{pw}(\phi^*, \widetilde{y})\widetilde{q}^d(\phi)^{\frac{\sigma-1}{\sigma}} \end{aligned} \tag{48}$$

and

$$\widetilde{r}_{\phi}(\phi) = (\sigma - 1)\,\widetilde{r}_{\phi}(\phi)\frac{\partial \ln A(\widetilde{H}(\phi),\phi)}{\partial \phi} \tag{49}$$

Noting that the third condition in (44) yields

$\frac{\sigma-1}{\sigma}\widetilde{q}^d(\phi)^{\frac{\sigma-1}{\sigma}}\left[1 + F(\widetilde{y}(\phi))\,\tau^{1-\sigma}n\right] g(\phi)\,\overline{M} = -h^H(\cdots,\phi)\lambda^H(\phi)$,

the fifth line in (44) implies

$$\begin{aligned} \widetilde{y}(\phi) &= \frac{\widetilde{r}^d(\phi)\,\tau^{1-\sigma}}{\sigma f_x C_0}, \\ C_0 &\equiv \tfrac{\sigma-1}{\sigma}\tfrac{\int_{\phi^*}^{\overline{\phi}} \widetilde{r}^d(\phi)\left[1+F(\widetilde{y}(\phi))\tau^{1-\sigma}n\right]g(\phi)\overline{M}d\phi}{L^{pw}(\phi^*,\widetilde{y})}. \end{aligned} \tag{50}$$

With the derivations above in mind, the first condition in (44) becomes

$$\widetilde{H}_{\phi}(\phi) = \frac{\widetilde{r}^d(\phi)\left[1 + F\left(\frac{\widetilde{r}^d(\phi)\tau^{1-\sigma}}{\sigma f_x C_0}\right)\tau^{1-\sigma}n\right] g(\phi)\,\overline{M}}{A(\widetilde{H}(\phi),\phi)V(\widetilde{H}(\phi))\widetilde{p}(\phi)\,L^{pw}(\phi^*, \frac{\widetilde{r}^d(\phi)\tau^{1-\sigma}}{\sigma f_x C_0})} \tag{51}$$

Finally, using the previous observations and

$\int_{\phi^*}^{\overline{\phi}} h^H_{\phi^*}(\cdots,\phi)\lambda^H(\phi)d\phi = -\int_{\phi^*}^{\overline{\phi}} h^H(\cdots,\phi)\lambda^H(\phi)\frac{fg(\phi^*)+nf_x\int_{y}^{\widetilde{y}(\phi^*)} ydF(y)g(\phi^*)\overline{M}}{L^{pw}(\phi^*,\frac{\widetilde{r}^d(\phi)\tau^{1-\sigma}}{\sigma f_x C_0})}d\phi$

in the last condition of (44) yields

$$\frac{\left[\sigma f + n\sigma f_x \int_y^{\frac{\widetilde{r}^d(\phi^*)\tau^{1-\sigma}}{\sigma f_x C_0}} ydF(y)\right]}{[1+F(\widetilde{y}(\phi^*))\tau^{1-\sigma}n]}\int_{\phi^*}^{\overline{\phi}} \frac{\widetilde{r}^d(\phi)}{\widetilde{r}^d(\phi^*)}\left[1 + F\left(\frac{\widetilde{r}^d(\phi)\tau^{1-\sigma}}{\sigma f_x C_0}\right)\tau^{1-\sigma}n\right] g(\phi)\,\overline{M}d\phi = \frac{\sigma}{\sigma-1}L^{pw}(\phi^*, \frac{\widetilde{r}^d(\phi)\tau^{1-\sigma}}{\sigma f_x C_0}).$$

If we choose the constant p_0 in (48) such that

$$\widetilde{r}^d(\phi^*) = \frac{\left[\sigma f + n\sigma f_x \int_y^{\frac{\widetilde{r}^d(\phi^*)\tau^{1-\sigma}}{\sigma f_x C_0}} y dF(y)\right]}{\left[1 + F\left(\frac{\widetilde{r}^d(\phi^*)\tau^{1-\sigma}}{\sigma f_x C_0}\right)\tau^{1-\sigma} n\right]}, \tag{52}$$

then the previous condition becomes

$$\int_{\phi^*}^{\overline{\phi}} \widetilde{r}^d(\phi)\left[1 + F\left(\tfrac{\widetilde{r}^d(\phi)\tau^{1-\sigma}}{\sigma f_x}\right)\tau^{1-\sigma} n\right] g(\phi)\overline{M} d\phi = \frac{\sigma}{\sigma - 1} L^{pw}(\phi^*, \tfrac{\widetilde{r}^d(\phi)\tau^{1-\sigma}}{\sigma f_x}) \tag{53}$$

as (50) yields $C_0 = 1$. Note that conditions (46), (49), (51),(53), $C_0 = 1$ and $\{\widetilde{H}(\phi^*) = \underline{s}, \widetilde{H}(\overline{\phi}) = \overline{s}\}$, imply that $\{\phi^*, \widetilde{p}, \widetilde{r}^d, \widetilde{H}\}$ satisfy all the conditions in lemma 3 with the exception of $\widetilde{r}^d(\phi^*) = \sigma f$. Accordingly, per condition (52), $\{\widetilde{p}, \widetilde{r}^d, \widetilde{H}\}$ is an equilibrium of the open economy only if $F\left(\frac{\widetilde{r}^d(\phi^*)\tau^{1-\sigma}}{\sigma f_x}\right) = \int_y^{\frac{\widetilde{r}^d(\phi^*)\tau^{1-\sigma}}{\sigma f_x C_0}} y dF(y) = 0$, a condition that is satisfied when the restriction on parameters assumed in the paper holds, $f\tau^{1-\sigma} \leq f_x$. As in the case of the closed economy, we can walk back on this derivations to show that given a triplet $\{p, r^d, H\}$ corresponding to an equilibrium of the open economy, then $\{q^d \equiv \frac{r^d}{p}, H\}$ is a solution to the planner's problem above when $f\tau^{1-\sigma} \leq f_x$.

When the restriction on parameters $f\tau^{1-\sigma} \leq f_x$ is not satisfied, the equivalence between equilibria of the open economy and solutions to problem (42) no longer holds. Intuitively, if $f\tau^{1-\sigma} > f_x$, then the planner is willing to accept some "negative domestic profits", $\widetilde{r}^d(\phi^*) < \sigma f$, because they are more than offset by positive export profits, $\widetilde{r}^d(\phi^*) F\left(\frac{\widetilde{r}^d(\phi^*)\tau^{1-\sigma}}{\sigma f_x}\right) n\tau^{1-\sigma} > \sigma f_x \int_y^{\frac{\widetilde{r}^d(\phi^*)\tau^{1-\sigma}}{\sigma f_x}} y dF(y)$. However, by changing slightly the arguments above, it can be shown that when $f\tau^{1-\sigma} > f_x$, the equilibria of the open economy are equivalent to solutions to constrained planner's problems that feature the following additional constraint

$$\sigma f \int_{\phi^*}^{\overline{\phi}} \left[\frac{\widetilde{q}^d(\phi)}{\widetilde{q}^d(\phi^*)}\right]^{\frac{\sigma-1}{\sigma}} \left[1 + F(\widetilde{y}(\phi))\tau^{1-\sigma} n\right] g(\phi)\overline{M} d\phi = \frac{\sigma}{\sigma - 1} L^{pw}(\phi^*, \widetilde{y}(\phi)).$$

Accordingly, $\pi$the equilibrium is constrained efficient in this case.

B.3 Additional Results related to BVP (20)

In this section, I present some results related to BVP (20) that are used in the text and in the proof of other results.

Lemma 4 *For $i = a, b$, let $\{z^i, x^i, \Gamma^i\}$ be the unique solution to the BVP (20) with parameters $\{\alpha^i(\phi), K_0^i, K_1^i\}$ and boundary conditions $x^i(\phi_0) = 1$, $\Gamma^i(\phi_0) = s_0$ and $\Gamma^i(\phi_1) = s_1$.*

(i) Suppose that $K_1^i = 0$, $\frac{\alpha^a(\phi')}{\alpha^a(\phi)} \geq \frac{\alpha^b(\phi')}{\alpha^b(\phi)}$ for all $\phi' > \phi \in [\phi_0, \phi_1]$, and $\frac{\alpha^a(\phi')}{\alpha^a(\phi)} > \frac{\alpha^b(\phi')}{\alpha^b(\phi)}$ for all $\phi' > \phi$ on some subinterval $[\phi_l, \phi_h] \subseteq [\phi_0, \phi_1]$. Then $\Gamma^a(\phi) < \Gamma^b(\phi)$ for all $\phi \in (\phi_0, \phi_1)$ and $\Gamma_\phi^a(\phi_0) < \Gamma_\phi^b(\phi_0)$ and

$\Gamma^a_\phi(\phi_1) > \Gamma^b_\phi(\phi_1)$.

(ii) Suppose that $K^i_0 = K_0$, $\alpha^i(\phi) = \alpha(\phi)$ *and* $K^b_1 < K^a_1$. *Then* $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$, *so there is a* $\phi^+ \in (\phi_0, \phi_1]$ *such that* $\Gamma^a(\phi^+) = \Gamma^b(\phi^+)$ *and* $\Gamma^a(\phi) < \Gamma^b(\phi)$ *for all* $\phi \in (\phi_0, \phi^+)$.

(iii) Let $\Phi^i \equiv \int_{\phi_0}^{\phi_1} x^i(\phi) \frac{\left[1+F\left(K^i_0 x^i(\phi)\right)K^i_1\right]}{\left[1+F\left(K^i_0\right)K^i_1\right]} \frac{\alpha^i(\phi)}{\alpha^i(\phi_0)} g(\phi)\, d\phi$. *If* $\Gamma^a(\phi) < \Gamma^b(\phi)$ *for* $\phi \in (\phi_0, \phi_1)$, *then* $\Phi^a > \Phi^b$.

(iv) If $\alpha^i(\phi) = \alpha(\phi)$, $K^b_0 = \lambda K^a_0$ *and* $K^b_1 = \lambda K^a_1$ *for* $\lambda > 1$, *then* $x^b(\phi)\lambda > x^a(\phi)$ *for all for all* $\phi \in [\phi_0, \phi_1]$.

(v) Let $\delta^i(\phi) \equiv \left[1 + F\left(K^i_0 x^i(\phi)\right) K^i_1\right] \alpha^i(\phi)$. *If* $\Gamma^a \neq \Gamma^b$ *and,* $\delta^a(\phi) < \delta^b(\phi)$ *for all* $\phi \in [\phi_0, \phi_1]$, *then*

$$\int_{\phi_0}^{\phi_1} x^a(\phi)\, \delta^a(\phi)\, g(\phi)\, d\phi < \int_{\phi_0}^{\phi_1} x^b(\phi)\, \delta^b(\phi)\, g(\phi)\, d\phi. \tag{54}$$

(vi) Suppose that $\left\{\alpha^i(\phi), K^i_1\right\} = \{\alpha(\phi), K_1\}$, $K^i_0, K_1 \in \mathbb{R}_{++}$ *and* $K^a_0 > K^b_0$. *If the function* $\eta_0(t, \lambda) \equiv \frac{F_y(t\lambda)\lambda K_1}{[1+F(t\lambda)K_1]}$ *is strictly decreasing (increasing) in* λ *on* $[1, \infty)$ *for* $t \in [K^b_0, K^b_0 x^b(\phi_1)]$, *then* $\Gamma^a(\phi) > (<)\Gamma^b(\phi)$ *on* (ϕ_0, ϕ_1), *with* $\Gamma^a_\phi(\phi_0) > (<)\Gamma^b_\phi(\phi)$.

(vii) Suppose that $\alpha^i(\phi) = \alpha(\phi)$, $K^i_0, K^i_1 \in \mathbb{R}_{++}$ *and* $K^a_i = \lambda K^b_i$ *for* $\lambda > 1$. *If the function* $\eta_1(t, \lambda) \equiv \frac{F_y(t\lambda)\lambda^2 K^b_1}{\left[1+F(t\lambda)\lambda K^b_1\right]}$ *is strictly increasing (decreasing) in* λ *on* $[1, \infty)$ *for* $t \in [K^b_0, K^b_0 x^b(\phi_1)]$, *then* $\Gamma^a(\phi) < (>)\Gamma^b(\phi)$ *on* (ϕ_0, ϕ_1) *with* $\Gamma^a_\phi(\phi_0) < (>)\Gamma^b_\phi(\phi_0)$.

Proof. Lemma 4.i. I proceed in steps.

STEP 1: *Under the assumptions of the lemma,* $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ *for all* $\phi \in (\phi_0, \phi_1)$.

Suppose to the contrary that there is a $\phi' \in (\phi_0, \phi_1)$ such that $\Gamma^a(\phi') > \Gamma^b(\phi')$. Let ϕ_- be the first time the functions Γ^a and Γ^b intersect to the left of ϕ' and let ϕ_+ be the first time they intersect to the right of ϕ'—i.e., $\phi_- \equiv \max\left\{\phi \leq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\right\}$ and $\phi_+ = \inf\left\{\phi \geq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\right\}$. Note that ϕ_- and ϕ_+ are well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the left and to the right of ϕ' (at ϕ_0 and at ϕ_1). Also note that $\Gamma^a(\phi) > \Gamma^b(\phi)$ for $\phi \in (\phi_-, \phi_+)$. The continuity of Γ^a_ϕ and Γ^b_ϕ, implies $\Gamma^a_\phi(\phi_-) \geq \Gamma^b_\phi(\phi_-)$ and $\Gamma^a_\phi(\phi_+) \leq \Gamma^b_\phi(\phi_+)$, so

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} \leq 1. \tag{55}$$

Differentiating the right-hand side of (24) yields

$$\frac{\Gamma^i_\phi(\phi_+)}{\Gamma^i_\phi(\phi_-)} = h^i(\phi_-, \phi_+)\, e^{\sigma \int_{\phi_-}^{\phi_+} \frac{\partial \ln A\left(\Gamma^i(u), u\right)}{\partial \phi} du}, \tag{56}$$

where $h^i(\phi_-, \phi_+)$ is given by (25) with $\alpha = \alpha^i$. By assumption, we have $\Gamma^a(\phi_-) = \Gamma^b(\phi_-)$ and $\Gamma^a(\phi_+) = \Gamma^b(\phi_+)$, which together with the definition of h^i, imply $\frac{h^a(\phi_-,\phi_+)}{h^b(\phi_-,\phi_+)} = \frac{\alpha^a(\phi_+)/\alpha^a(\phi_-)}{\alpha^b(\phi_+)/\alpha^b(\phi_-)}$. Combining this result and (56) yields

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} = e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial \ln A(\Gamma^a(u),u)}{\partial\phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial\phi}\right]du}\frac{\alpha^a(\phi_+)/\alpha^a(\phi_-)}{\alpha^b(\phi_+)/\alpha^b(\phi_-)}. \tag{57}$$

The strict log-supermodularity of A and the fact that $\Gamma^a(\phi) > \Gamma^b(\phi)$ for $\phi \in \left(\phi_-, \phi_+\right)$ imply that the first term of the last expression is strictly greater than 1. In addition, the assumption about relative values of α^a and α^b on $[\phi_0, \phi_1]$ implies that the second term is weakly greater than one, $\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} > 1$. This result contradicts (55), so it must be that $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ for $\phi \in [\phi_0, \phi_1]$.

STEP 2: *Under the assumptions in the lemma, $\Gamma^a(\phi)$ and $\Gamma^b(\phi)$ cannot satisfy $\Gamma^a(\phi) = \Gamma^b(\phi)$ on any nondegenerate interval $I \subseteq [\phi_l, \phi_h]$.*

Suppose to the contrary that $\Gamma^a(\phi) = \Gamma^b(\phi)$ for some nondegenerate interval $I \subseteq [\phi_l, \phi_h]$ and let $\phi_- < \phi_+$ be two interior points of I. Notice that $\Gamma^a(\phi) = \Gamma^b(\phi)$ on I implies that $\Gamma^a_\phi(\phi) = \Gamma^b_\phi(\phi)$ on the interior of I, so

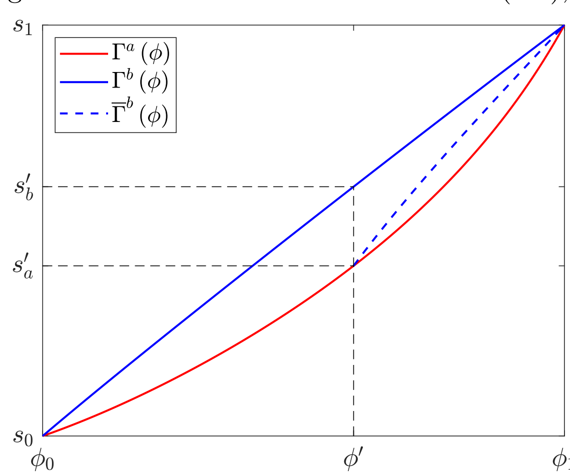
$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} = 1. \tag{58}$$

In addition, equation (57) must also hold in this case, which under the current assumptions yields

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_-)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_-)} = \frac{\alpha^a(\phi_+)/\alpha^a(\phi_-)}{\alpha^b(\phi_+)/\alpha^b(\phi_-)} > 1,$$

where the strict inequality follows from $\phi_-, \phi_+ \in [\phi_l, \phi_h]$ and the assumption about relative values of α^a and α^b on this interval. The last expression contradicts (58). Then it must be the case that Γ^a and Γ^b cannot be equal on any nondegenerate interval $I \subseteq [\phi_l, \phi_h]$.

Figure 9: Solutions to the General BVP (20), Γ



Note: The figure depicts solutions to alternative parametrizations of the general BVP (20). The BVPs corresponding to Γ^a and Γ^b differ only in the parameter function $\alpha(\phi)$ as indicated in lemma 4.i Restricted to $[\phi', \phi_1]$,the BVPs corresponding to Γ^b and $\overline{\Gamma}^b$ differ only in their initial conditions.

STEP 3: *Under the assumptions in the lemma,* $\Gamma^a(\phi) < \Gamma^b(\phi)$ for all $\phi \in (\phi_0, \phi_1)$

Steps 1 and 2 imply that there is a $\phi' \in (\phi_l, \phi_h) \subseteq [\phi_0, \phi_1]$ such that $\Gamma^a(\phi') < \Gamma^b(\phi')$. The situation is depicted in figure 9. Now, I prove that $\Gamma^a(\phi) < \Gamma^b(\phi)$ on $[\phi', \phi_1)$. To establish this result I show that there exists a function $\overline{\Gamma}^b : [\phi', \phi_1] \to S$ (dashed blue line), such that $\Gamma^a(\phi) \leq \overline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ for all $\phi \in [\phi', \phi_1)$. Letting $s_i' \equiv \Gamma^i(\phi')$ for $i = a, b$, if we define on $[\phi', \phi_1]$, $w^b(\phi) \equiv x^b(\phi)/x^b(\phi')$ and $y^b(\phi) \equiv z^b(\phi)/x^b(\phi')$, then $\{y^b, w^b, \Gamma^b\}$ is the unique solution to the BVP (20) on $[\phi', \phi_1]$ with parameters $\{\alpha^b(\phi), K_0^b, K_1^b\}$ and boundary conditions $w(\phi') = 1$, $\Gamma^b(\phi') = s_b'$ and $\Gamma^b(\phi_1) = s_1$.[54] Now, let $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ be the unique solution to the BVP (20) on $[\phi', \phi_1]$ with the same parameters and boundary conditions $\overline{x}^b(\phi') = 1$, $\overline{\Gamma}^b(\phi') = s_a' < s_b'$ and $\overline{\Gamma}^b(\phi_1) = s_1$. It is readily seen that $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ and $\{y^b, w^b, \Gamma^b\}$ satisfy the conditions of the no-crossing result in lemma 2.ii with $\overline{\Gamma}^b(\phi') < \Gamma^b(\phi')$, so $\overline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ on $[\phi', \phi_1)$. Defining w^a and y^a on $[\phi', \phi_1]$ from x^a and z^a as I did above implies that $\{y^a, w^a, \Gamma^a\}$ is the unique solution to the BVP (20) on $[\phi', \phi_1]$ with parameters $\{\alpha^a(\phi), K_0^a, K_1^a\}$ and boundary conditions $w^a(\phi') = 1$, $\Gamma^a(\phi') = s_a'$ and $\Gamma^a(\phi_1) = s_1$. Then, $\{w^a, y^a, \Gamma^a\}$ and $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ satisfy the conditions of step 1 above, so $\Gamma^a(\phi) \leq \overline{\Gamma}^b(\phi)$ on $[\phi', \phi_1]$ as depicted in the figure.

The argument in the last paragraph can be easily adapted to show that there is a function $\underline{\Gamma}^b : [\phi_0, \phi'] \to S$, such that $\Gamma^a(\phi) \leq \underline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ for all $\phi \in (\phi_0, \phi']$, completing the proof of step 3. Of note, this part of the argument requires slightly different version of the no-crossing result in proposition 2.ii. Specifically, in the notation of proposition 2, it can be shown that if we consider the solution to BVP (20) as a function of (s_0, s_1), then $\Gamma(\phi; s_0, s_1^a) < \Gamma(\phi; s_0, s_1^b)$ on $(\phi_0, \phi_1]$ if $s_1^a < s_1^b$.

STEP 4: *Under the same assumptions made in step 3,* $\Gamma_\phi^a(\phi_0) < \Gamma_\phi^b(\phi_0)$ *and* $\Gamma_\phi^a(\phi_1) > \Gamma_\phi^b(\phi_1)$.

Let $\phi' \in (\phi_0, \phi_1)$ and the triplets of functions $\{y^a, w^a, \Gamma^a\}$, $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ and $\{y^b, w^b, \Gamma^b\}$ on $[\phi', \phi_1]$ be defined as in step 3. Given that $\Gamma^a(\phi) \leq \overline{\Gamma}^b(\phi)$ on $[\phi', \phi_1]$, then it must be the case that $\Gamma_\phi^a(\phi_1) \geq \overline{\Gamma}_\phi^b(\phi_1)$, otherwise $\Gamma^a(\phi) > \overline{\Gamma}^b(\phi)$ on some neighborhood of ϕ_1. In a similar way, $\overline{\Gamma}^b(\phi) < \Gamma^b(\phi)$ on on $[\phi', \phi_1)$ implies $\overline{\Gamma}_\phi^b(\phi_1) \geq \Gamma_\phi^b(\phi_1)$. Moreover, if $\overline{\Gamma}_\phi^b(\phi_1) = \Gamma_\phi^b(\phi_1)$, then$\{\overline{y}^b, \overline{w}^b, \Gamma^b\}$—with $\overline{y}^b(\phi) = \frac{y^b(\phi)}{w^b(\phi_1)}\overline{x}^b(\phi_1)$ and $\overline{w}^b(\phi) = \frac{w^b(\phi)}{w^b(\phi_1)}\overline{x}^b(\phi_1)$—and $\{\overline{z}^b, \overline{x}^b, \overline{\Gamma}^b\}$ satisfy the same IVP with initial condition at ϕ_1, so $\overline{\Gamma}^b = \Gamma^b$ on $[\phi', \phi_1]$, contradicting our earlier results. Then it must be the case that $\overline{\Gamma}_\phi^b(\phi_1) > \Gamma_\phi^b(\phi_1)$. Putting together these results we get $\Gamma_\phi^a(\phi_1) \geq \overline{\Gamma}_\phi^b(\phi_1) > \Gamma_\phi^b(\phi_1)$. The other part of the claim can be proved making only minor adjustments to this argument.

Lemma 4.ii. I proceed in steps.

STEP 1: *Under the assumptions of the lemma, there is no* $\phi' \in (\phi_0, \phi_1]$ *such that* $\Gamma^a(\phi) \geq \Gamma^b(\phi)$ *for all* $\phi \in (\phi_0, \phi']$.

Suppose to the contrary that there is such a value $\phi' \in (\phi_0, \phi_1]$. Let ϕ_+ be the first time the functions Γ^a and Γ^b intersect to the right of ϕ'—i.e., $\phi_+ = \inf\{\phi \geq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\}$. Note ϕ_+ is well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the right of ϕ' (at ϕ_1). Also note that $\Gamma^a(\phi) \geq \Gamma^b(\phi)$ for $\phi \in (\phi_0, \phi_+)$. The continuity of Γ_ϕ^a and Γ_ϕ^b,

[54] Note that we are using the same notation to denote the restriction of a function to a subset of its domain.

implies $\Gamma^a_\phi(\phi_0) \geq \Gamma^b_\phi(\phi_0)$ and $\Gamma^a_\phi(\phi_+) \leq \Gamma^b_\phi(\phi_+)$, so

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} \leq 1. \tag{59}$$

Differentiating the right-hand side of (24) yields

$$\frac{\Gamma^i_\phi\left(\phi_+\right)}{\Gamma^i_\phi\left(\phi_0\right)} = h^i\left(\phi_0,\phi_+\right) e^{\sigma\int_{\phi_-}^{\phi_+}\frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial\phi}du} \frac{\left[1+F\left(K_0 x^i\left(\phi\right)\right)K^i_1\right]}{\left[1+F\left(K_0\right)K^i_1\right]}, \tag{60}$$

where $h^i\left(\phi_0,\phi_+\right)$ is given by (25). By assumption, we have $\Gamma^a\left(\phi_0\right) = \Gamma^b\left(\phi_0\right)$ and $\Gamma^a\left(\phi_+\right) = \Gamma^b\left(\phi_+\right)$, which together with the definition of h^i, imply $h^a\left(\phi_0,\phi_+\right) = h^b\left(\phi_0,\phi_+\right)$. Combining this result with (60) for $i = a, b$ yields

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} = e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial\ln A\left(\Gamma^a(u),u\right)}{\partial\phi}-\frac{\partial\ln A\left(\Gamma^b(u),u\right)}{\partial\phi}\right]du} \frac{\left[1+F\left(K_0x^a\left(\phi_+\right)\right)K^a_1\right]/\left[1+F\left(K_0\right)K^a_1\right]}{\left[1+F\left(K_0x^b\left(\phi_+\right)\right)K^b_1\right]/\left[1+F\left(K_0\right)K^b_1\right]}. \tag{61}$$

The strict log-supermodularity of A and the fact that $\Gamma^a\left(\phi\right) \geq \Gamma^b\left(\phi\right)$ for $\phi \in \left(\phi_0,\phi_+\right)$ imply that the first term of the right-hand side of the last expression is weakly greater than 1. In addition, note that we can write $\frac{\left[1+F\left(K_0x^i(\phi)\right)K^i_1\right]}{\left[1+F(K_0)K^i_1\right]} = \frac{1}{\left[1+F(K_0)K^i_1\right]} + \frac{F(K_0)K^i_1}{\left[1+F(K_0)K^i_1\right]}\frac{F\left(K_0x^i(\phi)\right)}{F(K_0)}$,
so $x^a\left(\phi\right) \geq x^b\left(\phi\right)$ for $\phi \in \left(\phi_0,\phi_+\right)$ $\left(\Gamma^a\left(\phi\right) \geq \Gamma^b\left(\phi\right)\right)$ and $K^a_1 > K^b_1$ imply that the second term of the right-hand side of (61) is strictly higher than one. Accordingly, $\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} > 1$, contradicting (59).

STEP 2: *Under the assumptions of the lemma, $\Gamma^a_\phi\left(\phi_0\right) < \Gamma^b_\phi\left(\phi_0\right)$, immediately proving the lemma.*

The result of step 1 immediately yields that $\Gamma^a_\phi\left(\phi_0\right) \leq \Gamma^b_\phi\left(\phi_0\right)$. Otherwise, $\Gamma^a_\phi\left(\phi_0\right) > \Gamma^b_\phi\left(\phi_0\right)$ implies that there is a $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a\left(\phi\right) > \Gamma^b\left(\phi\right)$ on $(\phi_0, \phi']$, contradicting the result in step 1. Suppose then that $\Gamma^a_\phi\left(\phi_0\right) = \Gamma^b_\phi\left(\phi_0\right) = \gamma_0$. Note that the (same) boundary conditions of the BVPs under consideration imply $\Gamma^i\left(\phi_0\right) = s_0$, $x^i\left(\phi_0\right) = 1$. In turn, these observations and equations (20a)-(20b) imply $x^i_\phi\left(\phi\right) = \frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}$ and $\frac{z^i_\phi(\phi_0)}{z^i(\phi_0)} = -\frac{\partial\ln A(s_0,\phi_0)}{\partial\phi}$. Log-differentiating both sides of equation (20c) and evaluating at ϕ_0 yields

$\frac{\Gamma^i_{\phi\phi}(\phi_0)}{\Gamma^i_\phi(\phi_0)} = \frac{x^i_\phi(\phi_0)}{x^i(\phi_0)} + \frac{F_y\left(K_0x^i(\phi_0)\right)K^i_1K_0x^i_\phi(\phi_0)}{\left[1+F(K_0x^i(\phi_0))K^i_1\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - [\frac{\partial\ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial s}\Gamma^i_\phi\left(\phi_0\right) + \frac{\partial\ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial\phi} + \frac{V_s\left(\Gamma^i(\phi_0)\right)}{V(\Gamma^i(\phi_0))}\Gamma^i_\phi\left(\phi_0\right) + \frac{z^i_\phi(\phi_0)}{z^i(\phi_0)}]$,

$\frac{\Gamma^i_{\phi\phi}(\phi)}{\gamma_0} = \frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi} + \frac{F_y(K_0)K^i_1K_0\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}}{\left[1+F(K_0)K^i_1\right]} + \frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)} + \frac{g_\phi(\phi_0)}{g(\phi_0)} - [\frac{\partial\ln A(s_0,\phi_0)}{\partial s}\gamma_0 + \frac{\partial\ln A(s_0,\phi_0)}{\partial\phi} + \frac{V_s(s_0)}{V(s_0)}\gamma_0 - \frac{\partial\ln A(s_0,\phi_0)}{\partial\phi}]$,

so,

$$\Gamma^a_{\phi\phi}\left(\phi_0\right) - \Gamma^b_{\phi\phi}\left(\phi_0\right) = \frac{F_y(K_0)K_0}{F(K_0)}\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}\gamma_0\left\{\frac{F(K_0)K^a_1}{\left[1+F(K_0)K^a_1\right]} - \frac{F(K_0)K^b_1}{\left[1+F(K_0)K^b_1\right]}\right\} > 0,$$

where the inequality follows from $K^a_1 > K^b_1$. The last expression implies that there is some $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a_\phi\left(\phi\right) > \Gamma^b_\phi\left(\phi\right)$ on $(\phi_0, \phi']$, which yields a contradiction of step 1. Accordingly, we must have

$\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$.

Finally, $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$ implies $\Gamma^a(\phi) < \Gamma^b(\phi)$ on some (small enough) interval (ϕ_0, ϕ''), so ϕ^+ described in the lemma is the first time Γ^a and Γ^b intersect to the right of ϕ''.

Lemma 4.iii.

The idea of the proof is to show that Γ^a and Γ^b can be thought of as the inverse of the matching functions of two artificial economies, and then use this additional information to prove the result. Let $\{z^i, x^i, \Gamma^i\}$ be the solution to the BVP in the statement of the lemma and consider the following artificial economy. In this economy there are no fixed costs of production and no fixed costs to export but the set of active firms and the set of exporters are fixed. In particular, the set of active firms are those with productivity in the range $[\phi_0, \phi_1]$, while the fraction of firms that export at each productivity level is given by $F(K_0^i x^i(\phi))$. The set of available workers are those with skills in the range $[s_0, s_1]$. The distribution of skills is given by the restriction of V to $[s_0, s_1]$ and the mass of workers is $\int_{s_0}^{s_1} V(s)\, ds L$. The total mass of firms with productivity ϕ is given by $g(\phi)\alpha^i(\phi)\overline{M}$, so the total mass of firms is $\int_{\phi_0}^{\phi_1} g(\phi)\alpha^i(\phi)\overline{M}$. Finally, τ_i is set to satisfy $K_1^i \equiv \tau_i^{1-\sigma}$.

Now I show that if p^i, $r^{d,i}$ and H^i denote the price, domestic revenue and inverse-matching functions of the economy described above, then $H^i = \Gamma^i$. An argument similar to the one in section 4 implies that $\{p^i, r^{d,i}, H^i\}$ satisfy the differential equations (11), (12) and

$$H^i_\phi(\phi) = \frac{r^{d,i}(\phi)\left[1 + F\left(K_0^i x^i(\phi)\right) K_1^i\right] g(\phi)\, \alpha^i(\phi)\, \overline{M}}{A\left(H^i(\phi), \phi\right) V\left(H^i(\phi)\right) p^i(\phi)\, L}, \tag{62}$$

with boundary conditions $H^i(\phi_0) = s_0$ and $H^i(\phi_1) = s_1$. Note that there is no boundary condition on the domestic revenue function $r^{d,i}$, as the zero-profit condition for firms with productivity ϕ_0 is no longer an equilibrium condition (no fixed costs of production). As a result, the levels of the functions $r^{d,i}$ and p^i cannot be determined without an additional condition (provided below). However, these conditions are enough to pin down H^i. To see this, let $\{p^i, r^{d,i}, H^i\}$ be any triplet of functions satisfying the equilibrium conditions described above, and define $\delta^i(\phi) \equiv \left[1 + F\left(K_0^i x^i(\phi)\right) K_1^i\right] \alpha^i(\phi)$, $v^i(\phi) \equiv r^{d,i}(\phi)/r^{d,i}(\phi_0)$ and $y^i(\phi) \equiv p^i(\phi)\, L / r^{d,i}(\phi_0)\, \overline{M}$. Then, it is readily seen that $\{y^i, v^i, H^i\}$ is the unique solution to the BVP (20) with parameter $K_1 = 0$ and $\alpha = \delta^i$.[55] However, note that, by construction, $\{z^i, x^i, \Gamma^i\}$ is also a solution to this parametrization of the BVP (20), so it must be the case that $H^i = \Gamma^i$.

Let us now derive an additional condition to pin down the revenue function of this artificial economy. In equilibrium, the total revenue of firms with productivity less or equal than ϕ' must equal a constant fraction of the total wages paid to workers employed at those firms,

$$r^{d,i}(\phi_0)\, \alpha^i(\phi_0) \left[1 + F\left(K_0^i\right) K_1^i\right] \int_{\phi_0}^{\phi'} x^i(\phi) \frac{\left[1 + F\left(K_0^i x^i(\phi)\right) K_1^i\right]}{\left[1 + F\left(K_0^i\right) K_1^i\right]} \frac{\alpha^i(\phi)}{\alpha^i(\phi_0)} g(\phi)\, \overline{M} d\phi = \tag{63}$$

$$\cdots \frac{\sigma}{\sigma - 1} L \int_{s_0}^{H^i(\phi')} w^i\left(H^i(\phi)\right) V\left(H^i(\phi)\right) ds, \text{ for } i = a, b.$$

[55] With $K_1 = 0$, the value of K_0 is irrelevant.

Differentiating the left- and right hand sides of the last expression with respect to ϕ', and evaluating the resulting expressions at $\phi' = \phi_0$ yields

$$r^{d,i}(\phi_0)\,\alpha^i(\phi_0)\left[1+F\left(K_0^i\right)K_1^i\right]g(\phi_0)\overline{M} = \frac{\sigma}{\sigma-1}Lw^i(s_0)\,V(s_0)\,H_\phi^i(\phi_0) \text{ for } i=a,b. \tag{64}$$

The last expression, together with the numeraire assumption, $\int_{s_0}^{s_1} w^i(s)\,V(s)\,ds = 1$, and the inverse matching function H^i, can be used to pin down the value of $r_d^i(\phi_0)$. To see this, note that H^i determines the growth rate of wages along the skill dimension (condition 10), while the numeraire assumption pins down their levels, so the wage schedule is fully determined. Then, equation (64) can be used to pin down $r_d^i(\phi_0)$, the only remaining endogenous variable.

With previous results we are ready to prove the lemma. As $H^a(\phi) < H^b(\phi)$ for $\phi \in [\phi_0, \phi_1]$ by assumption, wages grow faster along the skill dimension in economy a than in economy b, so the numeraire assumption implies $w^a(s_0) < w^b(s_0)$. In addition, $H^a(\phi) < H^b(\phi)$ for $\phi \in [\phi_0, \phi_1]$ also implies that $H_\phi^a(\phi_0) \leq H_\phi^b(\phi_0)$. These observations and (64) imply $r^{d,a}(\phi_0)\,\alpha^a(\phi_0)\left[1+F\left(K_0^a\right)K_1^a\right] < r^{d,b}(\phi_0)\,\alpha^b(\phi_0)\left[1+F\left(K_0^b\right)K_1^b\right]$. Finally, the last inequality, expression (63) evaluated at $\phi' = \phi_1$ for $i = a, b$, and the numeraire assumption yield the desired result.

Lemma 4.iv.

As Γ^i is a fixed point of the functional Ψ^i defined in (24) with parameters $\left\{\alpha^i(\phi), K_0^i, K_1^i\right\}$, $\Gamma^i(\phi) = \Psi^i\left(\Gamma^i\right)(\phi)$, $\Gamma_\phi^i(\phi)$ can be obtained differentiating the right-hand side of (24). Doing so yields,

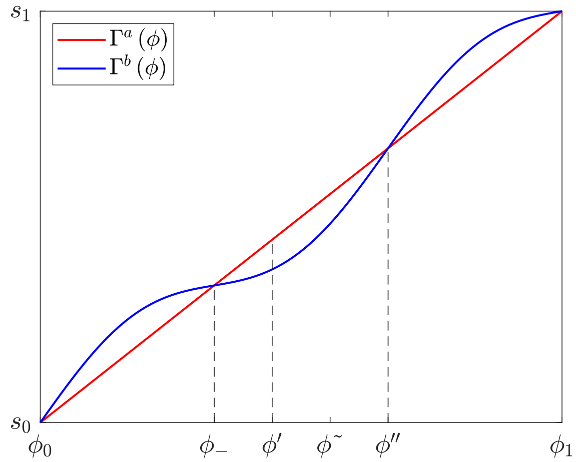
$$\Gamma_\phi^i(\phi) = [s_1 - s_0]\frac{h^i\left(\phi, \Gamma^i(\phi)\right)x^i(\phi)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^i x^i(\phi)\right)K_1^i\right]}{\int_{\phi_0}^{\phi_1} h^i\left(t, \Gamma^i(t)\right)x^i(t)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^i x^i(t)\right)K_1^i\right]dt}, \tag{65}$$

where in the last expression I used the fact that $x^i(\phi) = e^{(\sigma-1)\int_{\phi_0}^{\phi}\frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial\phi}du}$. The last expression plays a central role in the proof. Specifically, I show that if the claim of the lemma is not satisfied, then it is possible to derive contradicting implications regarding the values of the denominators on the right-hand side of (65), Dem^i for $i = a, b$. Throughout the proof, I denote the numerator on the right hand side of (65) by $Num^i(\phi)$.

Suppose the claim of the lemma is not true and $x^b(\phi)\lambda \leq x^a(\phi)$ for some $\phi \in [\phi_0, \phi_1]$. Noting that $x^a(\phi_0) < \lambda x^b(\phi_0)$, let $\phi^{\sim} > \phi_0$ be the lowest productivity value at which $x^b(\phi)\lambda = x^a(\phi)$.[56] Clearly, $x^a(\phi)$ must be catching up to $x^b(\phi)\lambda$ to the left of $\phi^{\sim}$, so equation (20b) implies that $\Gamma^b(\phi) < \Gamma^a(\phi)$ on some interval (ϕ', ϕ''), with $\phi' < \phi^{\sim} \leq \phi''$ and $\Gamma^b\left(\phi''\right) = \Gamma^a\left(\phi''\right)$. This situation is depicted in figure 10.

[56] Note that $\phi^{\sim}$ is well defined due to the continuity of the functions x^a and x^b.

Figure 10: Hypotetical Solutions to the General BVP (20), Γ



Note: The figure depicts hypothetical solutions to the general BVP (20) with the features implied by the assumption $x^b(\phi)\lambda \leq x^a(\phi)$ given the conditions in lemma 4.iv. as described in the proof. Of note, said assumption implies $\phi^{\sim} \in (\phi', \phi'']$, with the figure showing one of many possibilities.

By construction, $\Gamma^b_\phi(\phi'') \geq \Gamma^a_\phi(\phi'')$ and $x^b(\phi'')\lambda \leq x^a(\phi'')$, where the latter inequality implies

$$\begin{aligned} x^b(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^b x^b(\phi'')\right)K_1^b\right] &= x^b(\phi'')^{\frac{1}{\sigma-1}}\left[x^b(\phi'')+F\left(K_0^a\lambda x^b(\phi'')\right)x^b(\phi'')\lambda K_1^a\right] \\ &< x^a(\phi'')^{\frac{1}{\sigma-1}}\left[x^a(\phi'')+F\left(K_0^a x^a(\phi'')\right)x^a(\phi'')K_1^a\right] \\ &= x^a(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^a x^a(\phi'')\right)K_1^a\right]. \end{aligned} \tag{66}$$

In addition, by definition of h^i we have $h^a(\phi'', \Gamma^a(\phi'')) = h^b(\phi'', \Gamma^b(\phi''))$, which, together with the last expression, implies that $Num^b(\phi'') < Num^a(\phi'')$. This last result, $\Gamma^b_\phi(\phi'') \geq \Gamma^a_\phi(\phi'')$ and expression (65) yield $Dem^b < Dem^a$.

Expression (65) implies

$$\frac{\Gamma^b_\phi(\phi'')/\Gamma^b_\phi(\phi_0)}{\Gamma^a_\phi(\phi'')/\Gamma^a_\phi(\phi_0)} = \frac{x^b(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^b x^b(\phi'')\right)K_1^b\right]}{x^a(\phi'')^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^a x^a(\phi'')\right)K_1^a\right]}\frac{\left[1+F\left(K_0^a\right)K_1^a\right]}{\left[1+F\left(\lambda K_0^a\right)\lambda K_1^a\right]} < 1,$$

where the inequality follows from (66) and $\lambda > 1$. The last result and $\Gamma^b_\phi(\phi'') \geq \Gamma^a_\phi(\phi'')$ imply $\Gamma^b_\phi(\phi_0) > \Gamma^a_\phi(\phi_0)$, so $\Gamma^b(\phi) > \Gamma^a(\phi)$ on some neighborhood of ϕ_0 (excluding ϕ_0). Let ϕ_- be the lowest productivity value to the right of ϕ_0 such that $\Gamma^b(\phi^-) = \Gamma^a(\phi^-)$. As $\Gamma^b(\phi) > \Gamma^a(\phi)$ on (ϕ_0, ϕ_-), we have $\Gamma^b_\phi(\phi_-) \leq \Gamma^a_\phi(\phi_-)$ and $x^b(\phi_-) > x^a(\phi_-)$. Using these results and (65) yields

$$\frac{Dem^b}{Dem^a} = \frac{\Gamma^a_\phi(\phi^-)}{\Gamma^b_\phi(\phi^-)}\frac{x^b(\phi_-)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(\lambda K_0^a x^b(\phi_-)\right)\lambda K_1^a\right]}{x^a(\phi^-)^{\frac{\sigma}{\sigma-1}}\left[1+F\left(K_0^a x^a(\phi_-)\right)K_1^a\right]} > 1,$$

contradicting our previous finding, $Dem^b < Dem^a$. Then it must be the case that $x^b(\phi)\lambda > x^a(\phi)$ for all for all $\phi \in [\phi_0, \phi_1]$, which is the desired result.

Lemma 4.v.

As in the case of lemma 4.iii, the idea of the proof is to show that Γ^a and Γ^b can be thought of as the inverse matching functions of two artificial economies, and then use this additional information to prove the result. Moreover, I define these artificial economies here in the same way I did in the proof of lemma 4.iii. Let $\left\{z^i, x^i, \Gamma^i\right\}$ be the solution to the BVP in the statement of the lemma and consider the following artificial economy. In this economy there are no fixed costs of production and no fixed costs to export but the set of active firms and the set of exporters are fixed. In particular, the set of active firms are those with productivity in the range $[\phi_0, \phi_1]$, while the fraction of firms that export of each productivity level is given by $F\left(K_0^i x^i(\phi)\right)$. The set of available workers are those with skills in the range $[s_0, s_1]$. The distribution of skills is given by the restriction of V to $[s_0, s_1]$ and the mass of workers is $\int_{s_0}^{s_1} V(s)\, dsL$. The total mass of firms with productivity ϕ is given by $g(\phi)\alpha^i(\phi)\overline{M}$, so the total mass of firms is $\int_{\phi_0}^{\phi_1} g(\phi)\alpha^i(\phi)\overline{M}$. Finally, I set τ_i such that $K_1^i \equiv \tau_i^{1-\sigma}$.

The same argument used in the proof of lemma 4.iii implies that if p^i, $r^{d,i}$ and H^i are the price, domestic revenue and inverse-matching functions of the economy described above, then $H^i = \Gamma^i$. In addition, equation (63) also holds in this economy, which can be differentiated with respect to the limit of integration to get

$$r^{d,i}(\phi)\,\delta^i(\phi)\,g(\phi)\overline{M} = \frac{\sigma}{\sigma-1} L w^i\left(H^i(\phi)\right) V\left(H^i(\phi)\right) H_\phi^i(\phi) \text{ for } i = a, b, \tag{67}$$

where $\delta^i(\phi)$ was defined in the statement of the lemma. As discussed in the proof of lemma 4.iii, the last expression and the numeraire assumption, $\int_{s_0}^{s_1} w^i(s)\, V(s)\, ds = 1$, can be used to pin down the level of the domestic revenue function $r^{d,i}$. For this reason, the last expression is central in the proof of this lemma, as the main result is an immediate implication of the values recovered for $r^{d,i}(\phi_0)$ and equation (63).

STEP 1: *Let Φ^* be the set of productivity levels given by*

$$\Phi^* = \left\{\phi \in [\phi_0, \phi_1] : H^b(\phi) = H^a(\phi), H_\phi^b(\phi) \leq H_\phi^a(\phi)\right\},$$

and let S^* denote the set of corresponding skill levels, $S^* \equiv \left\{s \in [s_0, s_1] : s = H^i(\phi) \text{ for some } \phi \in \Phi^*\right\}$. Then, $w^b(s) < w^a(s)$ for some $s \in S^*$.

Suppose that this is not the case and $w^b(s) \geq w^a(s)$ for all $s \in S^*$ and let N^i be the matching function of the artificial economy described above, that is, N^i is the inverse function of H^i. For any $s \in [s_0, s_1] \setminus S^*$, there are three possibilities, (i) $N^a(s) = N^b(s)$, (ii) $N^a(s) < N^b(s)$, and (iii) $N^a(s) > N^b(s)$. I show that $w^b(s) > w^a(s)$ in all cases.

Let us start with case (i). As $s \notin S^*$, then $H_\phi^b(\phi) > H_\phi^a(\phi)$ for $\phi = N^i(s)$, implying $H^b\left(\phi'\right) < H^a\left(\phi'\right)$ on some neighborhood to the left of ϕ. Let ϕ^- be the first time H^a and H^b intersect to the left of ϕ, and let $s^- \equiv H^i\left(\phi^-\right)$. By construction, we have $\phi^- \in \Phi^*$ ($s^- \in S^*$) and $H^b(\phi') < H^a\left(\phi'\right)$ for all $\phi' \in \left(\phi^-, \phi\right)$

$(N^b(s') > N^a(s')$ for all $s' \in (s^-, s))$, so

$$w^b(s) = w^b\left(s^-\right) e^{\int_{s^-}^{s} \frac{\partial \ln A\left(t, N^b(t)\right)}{\partial s} dt} > w^a\left(s^-\right) e^{\int_{s_0}^{s} \frac{\partial \ln A\left(t, N^a(t)\right)}{\partial s} dt} = w^a(s), \tag{68}$$

where the last inequality in a consequence of the log-supermodularity of A and $w^b(s^-) \geq w^a(s^-)$.

Turning to case (ii), let s^- and s^+ be the first time N^a and N^b intersect to the left and right of s respectively. These skill levels are well defined due to the continuity of the functions involved and the fact that N^a and N^b intersect at least once to the left and right of s (at s_0 and s_1). Letting $\phi^k \equiv N^i\left(s^k\right)$ for $k = -, +$, by construction we have $N^b(s') > N^a(s')$ for all $s' \in (s^-, s^+)$, so $N_s^b(s^-) \geq N_s^a(s^-)$ $(H_\phi^b\left(\phi^-\right) \leq H_\phi^a\left(\phi^-\right))$—i.e., $s^- \in S^*$. Then inequality (68) also holds in this case.

Let us now turn to case (iii). Let s^- and s^+ be the first time N^a and N^b intersect to the left and right of s respectively. As before, these skill levels are well defined. Letting $\phi^k \equiv N^i\left(s^k\right)$ for $k = -, +$, by construction we have $N^b(s') < N^a(s')$ for all $s' \in (s^-, s^+)$, so $N_s^b(s^+) \geq N_s^a(s^+)$ $(H_\phi^b\left(\phi^+\right) \leq H_\phi^a\left(\phi^+\right))$—i.e., $s^+ \in S^*$. This and the log supermodularity of A imply

$$\frac{w^b(s^+)}{w^b(s)} = e^{\int_{s}^{s^+} \frac{\partial \ln A\left(t, N^b(t)\right)}{\partial s} dt} < e^{\int_{s}^{s^+} \frac{\partial \ln A\left(t, N^a(t)\right)}{\partial s} dt} = \frac{w^a(s^+)}{w^a(s)}.$$

Per our initial assumption and $s^+ \in S^*$ we have $w^b(s^+) \geq w^a(s^+)$, which together with the last expression, yields $w^b(s) > w^a(s)$.

Given that the selection of $s \in [s_0, s_1] \backslash S^*$ was arbitrary, we conclude that $w^b(s) > w^a(s)$ for all $s \in [s_0, s_1] \backslash S^*$. However, notice that $w^b(s) \geq w^a(s)$ on $[s_0, s_1]$ and $w^b(s) > w^a(s)$ on $[s_0, s_1] \backslash S^*$ imply $\overline{w}^b > \overline{w}^a$, which contradicts our numeraire selection. Then it must be the case that $w^b(s) < w^a(s)$ for some $s \in S^*$.

STEP 2: *Let S^* be defined as before, let $s^+ \in S^*$ such that $w^b(s^+) < w^a(s^+)$ and let $\phi^+ = N^i(s^+)$. If $x^b\left(\phi^+\right) \geq x^a\left(\phi^+\right)$, then $r^{d,b}(\phi_0) < r^{d,a}(\phi_0)$.*

By assumption we have $w^b(s^+) < w^a(s^+)$, $H^a\left(\phi^+\right) = H^b\left(\phi^+\right)$ and $H_\phi^b\left(\phi^+\right) \leq H_\phi^b\left(\phi^+\right)$, which, together with equation (67) evaluated at ϕ^+, imply

$$r^{d,b}(\phi_0)\, x^b\left(\phi^+\right) \delta^b\left(\phi^+\right) < r^{d,a}(\phi_0)\, x^a\left(\phi^+\right) \delta^a\left(\phi^+\right).$$

The last expression, $x^b\left(\phi^+\right) \geq x^a\left(\phi^+\right)$, and the assumption in the of the lemma $(\delta^b(\phi) > \delta^a(\phi))$ imply $r^{d,b}(\phi_0) < r^{d,a}(\phi_0)$.

STEP 3: *Let S^*, s^+ and ϕ^+ be defined as in step 2. If $x^b\left(\phi^+\right) < x^a\left(\phi^+\right)$, then $r^{d,b}(\phi_0) < r^{d,a}(\phi_0)$.*

The continuity of H^i and of H^i_ϕ imply that Φ^* and S^* are closed sets, so let $s^- \equiv \inf S^* \in S^*$. As H^a and H^b intersect at ϕ_0 and at ϕ^+, the following equality holds for $i = a, b$,

$$\begin{aligned}\ln \frac{A\left(s^+, \phi^+\right)}{A\left(s_0, \phi_0\right)} &= \int_{\phi_0}^{\phi^+} \frac{\partial \ln A\big(H^i(t),t\big)}{\partial s} H^i_\phi(t)dt + \int_{\phi_0}^{\phi^+} \frac{\partial \ln A\big(H^i(t),t\big)}{\partial \phi} dt \\ &= \int_{s_0}^{s^+} \frac{\partial \ln A\big(u,N^i(u)\big)}{\partial s} du + \int_{\phi_0}^{\phi^+} \frac{\partial \ln A\big(H^i(t),t\big)}{\partial \phi} dt.\end{aligned}$$

Note that the second term in the right-hand side of the last expression is proportional to $\ln x^i\left(\phi^+\right)$. As such, the last expression and the assumption $x^b\left(\phi^+\right) < x^a\left(\phi^+\right)$ yields $\int_{s_0}^{s^+} \frac{\partial \ln A\left(u,N^b(u)\right)}{\partial s} du > \int_{s_0}^{s^+} \frac{\partial \ln A\left(u,N^a(u)\right)}{\partial s} du$, which, together with condition (10), implies

$$\frac{w^b\left(s^-\right)}{w^b\left(s_0\right)} \frac{w^b\left(s^+\right)}{w^b\left(s^-\right)} = \int_{s_0}^{s^+} \frac{\partial \ln A\big(u,N^b(u)\big)}{\partial s} du > \int_{s_0}^{s^+} \frac{\partial \ln A(u,N^a(u))}{\partial s} du = \frac{w^a\left(s^-\right)}{w^a\left(s_0\right)} \frac{w^a\left(s^+\right)}{w^a\left(s^-\right)}. \tag{69}$$

Now I show that $w^b\left(s^-\right)/w^b\left(s_0\right) \leq w^a\left(s^-\right)/w^a\left(s_0\right)$. If $s^- = s_0$ there is nothing to prove, so let's assume that $s^- > s_0$. First, notice that $N^b\left(s\right) \leq N^a\left(s\right)$ for $s \in [s_0, s^-]$. To see this, suppose to the contrary that $N^b\left(s\right) > N^a\left(s\right)$ for some $s \in (s_0, s^-)$, and let s' be the first time N^b and N^a intersect to the left of s. Then we have $s' < s^-$, $N^b\left(s'\right) = N^a\left(s'\right)$ and $N^b_s\left(s'\right) \geq N^a_s\left(s'\right)$—i.e., $s' \in s^*$ with $s' < s^-$. However, this contradicts the definition of s^-, so it must be the case that $N^b\left(s\right) \leq N^a\left(s\right)$ for $s \in [s_0, s^-]$. This result and the log supermodularity of A implies

$$\frac{w^b\left(s^-\right)}{w^b\left(s_0\right)} = \int_{s_0}^{s^-} \frac{\partial \ln A\big(u,N^b(u)\big)}{\partial s} du \leq \int_{s_0}^{s^-} \frac{\partial \ln A(u,N^a(u))}{\partial s} du = \frac{w^a\left(s^-\right)}{w^a\left(s_0\right)}. \tag{70}$$

The inequalities (69)-(70) and our assumption $w^b\left(s^+\right) < w^a\left(s^+\right)$ imply $w^b\left(s^-\right) < w^a\left(s^-\right)$. Using this result, $H^b_\phi(\phi^-) \leq H^a_\phi(\phi^-)$ and the assumption in the lemma about $\delta^i\left(\phi\right)$ in expression (67) (evaluated at ϕ^-) yields $r^{d,b}\left(\phi^-\right) < r^{d,a}\left(\phi^-\right)$. If $\phi^- = \phi_0$, we are done, so let us assume $\phi^- > \phi_0$. As discussed above, $N^b\left(s\right) \leq N^a\left(s\right)$ for $s \in [s_0, s^-]$ ($H^b\left(\phi\right) \geq H^a\left(\phi\right)$ for $\phi \in \left[\phi_0, \phi^-\right]$), implying

$$\frac{r^{d,b}\left(\phi^-\right)}{r^{d,b}\left(\phi_0\right)} = e^{(\sigma-1)\int_{\phi_0}^{\phi^-} \frac{\partial \ln A(H^b(t),t)}{\partial \phi} dt} \geq e^{(\sigma-1)\int_{\phi_0}^{\phi^-} \frac{\partial \ln A(H^a(t),t)}{\partial \phi} dt} = \frac{r^{d,a}\left(\phi^-\right)}{r^{d,a}\left(\phi_0\right)}.$$

The last expression and $r^{d,b}\left(\phi^-\right) < r^{d,a}\left(\phi^-\right)$ imply $r^{d,b}\left(\phi_0\right) < r^{d,a}\left(\phi_0\right)$, which is the desired result.

STEP 4: *Under the assumptions of the Lemma, inequality (54) holds.*

Steps 2 and 3 together imply that $r^{d,b}\left(\phi_0\right) < r^{d,a}\left(\phi_0\right)$, holds for these two artificial economies. This result, the numeraire assumption for these economies and equation (63) evaluated at $\phi' = \phi_1$ imply that inequality (54) holds.

Lemma 4.vi. I prove the statement for the case in which $\eta_0\left(t, \lambda\right)$ is strictly decreasing in λ.

STEP 1: *Under the assumptions of the lemma, there is no $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a\left(\phi\right) \leq \Gamma^b\left(\phi\right)$ for all $\phi \in (\phi_0, \phi']$.*

Suppose to the contrary that there is such a value $\phi' \in (\phi_0, \phi_1]$. Let ϕ_+ be the first time the functions Γ^a and Γ^b intersect to the right of ϕ'—i.e., $\phi_+ = \inf\left\{\phi \geq \phi' : \Gamma^a(\phi) = \Gamma^b(\phi)\right\}$. Note ϕ_+ is well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the right of ϕ' (at ϕ_1). Also note that $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ for $\phi \in (\phi_0, \phi_+)$. The continuity of Γ^a_ϕ and Γ^b_ϕ, implies $\Gamma^a_\phi(\phi_0) \leq \Gamma^b_\phi(\phi_0)$ and $\Gamma^a_\phi(\phi_+) \geq \Gamma^b_\phi(\phi_+)$, so

$$\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} \geq 1. \tag{71}$$

Differentiating the right-hand side of (24) yields

$$\frac{\Gamma^i_\phi(\phi_+)}{\Gamma^i_\phi(\phi_0)} = h^i(\phi_0, \phi_+)\, e^{\sigma \int_{\phi_-}^{\phi_+} \frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial \phi} du} \frac{\left[1 + F\left(K^i_0 x^i(\phi)\right) K_1\right]}{\left[1 + F\left(K^i_0\right) K_1\right]}, \tag{72}$$

where $h^i(\phi_0, \phi_+)$ is given by (25). By assumption, we have $\Gamma^a(\phi_0) = \Gamma^b(\phi_0)$ and $\Gamma^a(\phi_+) = \Gamma^b(\phi_+)$, which together with the definition of h^i, imply $h^a(\phi_0, \phi_+) = h^b(\phi_0, \phi_+)$. Combining this result with (72) for $i = a, b$ yields

$$\begin{aligned}
\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} &= e^{\sigma \int_{\phi_-}^{\phi_+} \left[\frac{\partial \ln A\left(\Gamma^a(u),u\right)}{\partial \phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial \phi}\right] du} \frac{\left[1+F\left(K^a_0 x^a(\phi_+)\right)K_1\right]/\left[1+F\left(K^a_0\right)K_1\right]}{\left[1+F\left(K^b_0 x^b(\phi_+)\right)K_1\right]/\left[1+F\left(K^b_0\right)K_1\right]}, \\
\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)} &\leq e^{\sigma \int_{\phi_-}^{\phi_+} \left[\frac{\partial \ln A\left(\Gamma^a(u),u\right)}{\partial \phi} - \frac{\partial \ln A\left(\Gamma^b(u),u\right)}{\partial \phi}\right] du} \frac{\left[1+F\left(K^b_0 \lambda x^b(\phi_+)\right)K_1\right]/\left[1+F\left(K^b_0 \lambda\right)K_1\right]}{\left[1+F\left(K^b_0 x^b(\phi_+)\right)K_1\right]/\left[1+F\left(K^b_0\right)K_1\right]},
\end{aligned} \tag{73}$$

where the second line uses $\lambda \equiv K^a_0/K^b_0 > 1$ and $x^b(\phi_+) \geq x^a(\phi_+)$, with the latter being a consequence of the strict log-supermodularity of A and the fact that $\Gamma^a(\phi) \leq \Gamma^b(\phi)$ for $\phi \in (\phi_0, \phi_+)$. Another implication of this last observation is that the first term of the right-hand side of the last expression is weakly lower than 1. Focusing on the second term, note that

$$\begin{aligned}
\frac{\left[1+F\left(K^b_0 \lambda x^b(\phi_+)\right)K_1\right]}{\left[1+F\left(K^b_0 \lambda\right)K_1\right]} &= \exp\{\int_{\phi_0}^{\phi_+} \frac{F_y\left(K^b_0 \lambda x^b(\phi)\right)K^b_0 K_1 \lambda x^b_\phi(\phi)}{\left[1+F\left(K^b_0 \lambda x^b(\phi)\right)K_1\right]} d\phi\} = \exp\{\int_{\phi_0}^{\phi_+} \eta^0\left(K^b_0 x^b(\phi), \lambda\right) K^b_0 x^b_\phi(\phi)\, d\phi\} \\
\frac{\left[1+F\left(K^b_0 x^b(\phi_+)\right)K_1\right]}{\left[1+F\left(K^b_0\right)K_1\right]} &= \exp\{\int_{\phi_0}^{\phi_+} \frac{F_y\left(K^b_0 x^b(\phi)\right)K_1 K^b_0 x^b_\phi(\phi)}{\left[1+F\left(K^b_0 x^b(\phi)\right)K_1\right]} d\phi\} = \exp\{\int_{\phi_0}^{\phi_+} \eta^0\left(K^b_0 x^b(\phi), 1\right) K^b_0 x^b_\phi(\phi)\, d\phi\}
\end{aligned} \tag{74}$$

As η^0 is strictly decreasing in λ, the second line in (74) is strictly greater than the first, so the second term on the right-hand side of the second line of (73) is strictly lower than 1, contradicting (71). Accordingly, the statement in step 1 must be true.

STEP 2: *Under the assumptions of the lemma, $\Gamma^a_\phi(\phi_0) > \Gamma^b_\phi(\phi_0)$, so there is a $\phi_+ \in (\phi_0, \phi_1)$ such that $\Gamma^a(\phi_+) = \Gamma^b(\phi_+)$ and $\Gamma^a(\phi) > \Gamma^b(\phi)$ on (ϕ_0, ϕ_+).*

The result of step 1 immediately yields that $\Gamma^a_\phi(\phi_0) \geq \Gamma^b_\phi(\phi_0)$. Otherwise, $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$ implies that there is a $\phi' \in (\phi_0, \phi_1]$ such that $\Gamma^a(\phi) < \Gamma^b(\phi)$ on $(\phi_0, \phi']$, contradicting the result in step 1. Suppose then that $\Gamma^a_\phi(\phi_0) = \Gamma^b_\phi(\phi_0) = \gamma_0$. Note that the (same) boundary conditions of the BVPs under consideration imply $\Gamma^i(\phi_0) = s_0$, $x^i(\phi_0) = 1$. In turn, these observations and equations (20a)-(20b) imply

$x_{\phi}^{i}(\phi) = \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi}$ and $\frac{z_{\phi}^{i}(\phi_0)}{z^{i}(\phi_0)} = -\frac{\partial \ln A(s_0,\phi_0)}{\partial \phi}$. Log-differentiating both sides of equation (20c) and evaluating at ϕ_0 yields

$\frac{\Gamma_{\phi\phi}^{i}(\phi_0)}{\Gamma_{\phi}^{i}(\phi_0)} = \frac{x_{\phi}^{i}(\phi_0)}{x^{i}(\phi_0)} + \frac{F_y\left(K_0^i x^i(\phi_0)\right)K_1 K_0^i x_{\phi}^{i}(\phi_0)}{\left[1+F\left(K_0^i x^i(\phi_0)\right)K_1\right]} + \frac{\alpha_{\phi}(\phi_0)}{\alpha(\phi_0)} + \frac{g_{\phi}(\phi_0)}{g(\phi_0)} - [\frac{\partial \ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial s}\Gamma_{\phi}^{i}(\phi_0) + \frac{\partial \ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial \phi} + \frac{V_s\left(\Gamma^i(\phi_0)\right)}{V\left(\Gamma^i(\phi_0)\right)}\Gamma_{\phi}^{i}(\phi_0) + \frac{z_{\phi}^{i}(\phi_0)}{z^{i}(\phi_0)}]$,

$\frac{\Gamma_{\phi\phi}^{i}(\phi)}{\gamma_0} = \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi} + \frac{F_y\left(K_0^i\right)K_1 K_0^i \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi}}{\left[1+F\left(K_0^i\right)K_1\right]} + \frac{\alpha_{\phi}(\phi_0)}{\alpha(\phi_0)} + \frac{g_{\phi}(\phi_0)}{g(\phi_0)} - [\frac{\partial \ln A(s_0,\phi_0)}{\partial s}\gamma_0 + \frac{\partial \ln A(s_0,\phi_0)}{\partial \phi} + \frac{V_s(s_0)}{V(s_0)}\gamma_0 - \frac{\partial \ln A(s_0,\phi_0)}{\partial \phi}]$,

so,

$$\Gamma_{\phi\phi}^{a}(\phi_0) - \Gamma_{\phi\phi}^{b}(\phi_0) = K_0^b \frac{(\sigma-1)\partial \ln A(s_0,\phi_0)}{\partial \phi}\gamma_0 \left\{ \frac{F_y\left(K_0^b\lambda\right)\lambda K_1}{\left[1+F\left(K_0^b\lambda\right)K_1\right]} - \frac{F_y\left(K_0^b\right)K_1}{\left[1+F\left(K_0^b\right)K_1\right]} \right\} < 0,$$

where the inequality follows from $\eta^0\left(K_0^b,\lambda\right) < \eta^0\left(K_0^b,1\right)$. The last expression implies that there is some $\phi' \in (\phi_0,\phi_1]$ such that $\Gamma_{\phi}^{a}(\phi) < \Gamma_{\phi}^{b}(\phi)$ on $(\phi_0,\phi']$, which yields a contradiction of step 1. Accordingly, we must have $\Gamma_{\phi}^{a}(\phi_0) > \Gamma_{\phi}^{b}(\phi_0)$.

Finally, $\Gamma_{\phi}^{a}(\phi_0) > \Gamma_{\phi}^{b}(\phi_0)$ implies $\Gamma^{a}(\phi) > \Gamma^{b}(\phi)$ on some (small enough) interval $\left(\phi_0,\phi''\right)$, so ϕ^{+} described in the statement of the step the first time Γ^a and Γ^b intersect to the right of ϕ''.

STEP 3: *Under the assumptions of the lemma,* $\Gamma^{a}(\phi) > \Gamma^{b}(\phi)$ *on* (ϕ_0,ϕ_1).

I show that $\phi_+ = \phi_1$, where ϕ_+ was defined in step 2. Suppose for a moment that $\phi_+ < \phi_1$. If we define on $[\phi_+,\phi_1]$, $w^i(\phi) \equiv x^i(\phi)/x^i\left(\phi_+\right)$ and $y^i(\phi) = z^i(\phi)/x^i\left(\phi_+\right)$, then it is readily seen that $\left\{y^i, w^i(\phi), \Gamma^i\right\}$ solve BVP (20) in said interval, with $\left\{\alpha^i(\phi), K_1^i\right\} = \left\{\alpha(\phi), K_1\right\}$ and parameter $\overline{K}_0^i = K_0^i x^i\left(\phi_+\right)$. Per step 2 we have $x^a\left(\phi_+\right) > x^b\left(\phi_+\right)$, so $\overline{K}_0^a > \overline{K}_0^b$. Then, the BVPs associated to $\left\{y^i, w^i(\phi), \Gamma^i\right\}$ satisfy the conditions of lemma 4.vi, so step 2 implies $\Gamma_{\phi}^{a}\left(\phi_+\right) > \Gamma_{\phi}^{b}\left(\phi_+\right)$. However, $\Gamma^{a}(\phi) > \Gamma^{b}(\phi)$ on $\left(\phi_0,\phi_+\right)$ implies $\Gamma_{\phi}^{a}\left(\phi_+\right) \leq \Gamma_{\phi}^{b}\left(\phi_+\right)$, so assuming $\phi_+ < \phi_1$ yields a contradiction.

Lemma 4.vii. I prove the statement for the case in which $\eta_1(t,\lambda)$ is strictly increasing in λ.

STEP 1: *Under the assumptions of the lemma, there is no* $\phi' \in (\phi_0,\phi_1]$ *such that* $\Gamma^{a}(\phi) \geq \Gamma^{b}(\phi)$ *for all* $\phi \in (\phi_0,\phi']$.

Suppose to the contrary that there is such a value $\phi' \in (\phi_0,\phi_1]$. Let ϕ_+ be the first time the functions Γ^a and Γ^b intersect to the right of ϕ', that is, $\phi_+ = \inf\left\{\phi \geq \phi' : \Gamma^{a}(\phi) = \Gamma^{b}(\phi)\right\}$. Note ϕ_+ is well defined due to the continuity of the functions Γ^a and Γ^b and the fact that the functions intersect at least once to the right of ϕ' (at ϕ_1). Also note that $\Gamma^{a}(\phi) \geq \Gamma^{b}(\phi)$ for $\phi \in \left(\phi_0,\phi_+\right)$. The continuity of Γ_{ϕ}^{a} and Γ_{ϕ}^{b}, implies $\Gamma_{\phi}^{a}(\phi_0) \geq \Gamma_{\phi}^{b}(\phi_0)$ and $\Gamma_{\phi}^{a}(\phi_+) \leq \Gamma_{\phi}^{b}(\phi_+)$, so

$$\frac{\Gamma_{\phi}^{a}(\phi_+)/\Gamma_{\phi}^{a}(\phi_0)}{\Gamma_{\phi}^{b}(\phi_+)/\Gamma_{\phi}^{b}(\phi_0)} \leq 1. \tag{75}$$

Differentiating the right-hand side of (24) yields

$$\frac{\Gamma_{\phi}^{i}\left(\phi_+\right)}{\Gamma_{\phi}^{i}\left(\phi_0\right)} = h^i\left(\phi_0,\phi_+\right) e^{\sigma \int_{\phi_-}^{\phi_+} \frac{\partial \ln A\left(\Gamma^i(u),u\right)}{\partial \phi} du} \frac{\left[1 + F\left(K_0^i x^i(\phi)\right)K_1^i\right]}{\left[1 + F\left(K_0^i\right)K_1^i\right]}, \tag{76}$$

where $h^i\left(\phi_0,\phi_+\right)$ is given by (25). By assumption, we have $\Gamma^a\left(\phi_0\right)=\Gamma^b\left(\phi_0\right)$ and $\Gamma^a\left(\phi_+\right)=\Gamma^b\left(\phi_+\right)$, which together with the definition of h^i, imply $h^a\left(\phi_0,\phi_+\right)=h^b\left(\phi_0,\phi_+\right)$. Combining this result with (76) for $i=a,b$ yields

$$\begin{aligned}\frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)}&=e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial\ln A\left(\Gamma^a(u),u\right)}{\partial\phi}-\frac{\partial\ln A\left(\Gamma^b(u),u\right)}{\partial\phi}\right]du}\frac{\left[1+F\left(K_0^ax^a\left(\phi_+\right)\right)K_1^a\right]/\left[1+F\left(K_0^a\right)K_1^a\right]}{\left[1+F\left(K_0^bx^b\left(\phi_+\right)\right)K_1^b\right]/\left[1+F\left(K_0^b\right)K_1^b\right]},\\ \frac{\Gamma^a_\phi(\phi_+)/\Gamma^a_\phi(\phi_0)}{\Gamma^b_\phi(\phi_+)/\Gamma^b_\phi(\phi_0)}&\geq e^{\sigma\int_{\phi_-}^{\phi_+}\left[\frac{\partial\ln A\left(\Gamma^a(u),u\right)}{\partial\phi}-\frac{\partial\ln A\left(\Gamma^b(u),u\right)}{\partial\phi}\right]du}\frac{\left[1+F\left(K_0^b\lambda x^b\left(\phi_+\right)\right)K_1^b\lambda\right]/\left[1+F\left(K_0^b\lambda\right)K_1^b\lambda\right]}{\left[1+F\left(K_0^bx^b\left(\phi_+\right)\right)K_1^b\right]/\left[1+F\left(K_0^b\right)K_1^b\right]},\end{aligned}\tag{77}$$

where the second line uses $\lambda\equiv K_i^a/K_i^b>1$ and $x^a\left(\phi_+\right)\geq x^b\left(\phi_+\right)$, with the latter being a consequence of the strict log-supermodularity of A and the fact that $\Gamma^a\left(\phi\right)\geq\Gamma^b\left(\phi\right)$ for $\phi\in\left(\phi_0,\phi_+\right)$. Another implication of the last observation is that the first term of the right-hand side of the last expression is weakly greater than 1. Focusing on the second term, note that

$$\begin{aligned}\frac{\left[1+F\left(K_0^b\lambda x^b\left(\phi_+\right)\right)K_1^b\lambda\right]}{\left[1+F\left(K_0^b\lambda\right)K_1^b\lambda\right]}&=\exp\{\int_{\phi_0}^{\phi_+}\frac{F_y\left(K_0^b\lambda x^b(\phi)\right)K_0^bK_1^b\lambda^2x^b_\phi(\phi)}{\left[1+F\left(K_0^b\lambda x^b(\phi)\right)K_1^b\lambda\right]}d\phi\}=\exp\{\int_{\phi_0}^{\phi_+}\eta^1\left(K_0^bx^b\left(\phi\right),\lambda\right)K_0^bx^b_\phi\left(\phi\right)d\phi\}\\ \frac{\left[1+F\left(K_0^bx^b\left(\phi_+\right)\right)K_1^b\right]}{\left[1+F\left(K_0^b\right)K_1^b\right]}&=\exp\{\int_{\phi_0}^{\phi_+}\frac{F_y\left(K_0^bx^b(\phi)\right)K_0^bK_1^bx^b_\phi(\phi)}{\left[1+F\left(K_0^bx^b(\phi)\right)K_1^b\right]}d\phi\}=\exp\{\int_{\phi_0}^{\phi_+}\eta^1\left(K_0^bx^b\left(\phi\right),1\right)K_0^bx^b_\phi\left(\phi\right)d\phi\}\end{aligned}\tag{78}$$

As η^1 is strictly increasing in λ, the second line in (78) is strictly lower than the first, so the second term on the right-hand side of the second line of (77) is strictly greater than 1, contradicting (75). Accordingly, the statement in step 1 must be true.

STEP 2: *Under the assumptions of the lemma, $\Gamma^a_\phi\left(\phi_0\right)<\Gamma^b_\phi\left(\phi_0\right)$, so there is a $\phi_+\in\left(\phi_0,\phi_1\right)$ such that $\Gamma^a\left(\phi_+\right)=\Gamma^b\left(\phi_+\right)$ and $\Gamma^a\left(\phi\right)<\Gamma^b\left(\phi\right)$ on $\left(\phi_0,\phi_+\right)$.*

The result of step 1 immediately yields that $\Gamma^a_\phi\left(\phi_0\right)\leq\Gamma^b_\phi\left(\phi_0\right)$. Otherwise, $\Gamma^a_\phi\left(\phi_0\right)>\Gamma^b_\phi\left(\phi_0\right)$ implies that there is a $\phi'\in\left(\phi_0,\phi_1\right]$ such that $\Gamma^a\left(\phi\right)>\Gamma^b\left(\phi\right)$ on $\left(\phi_0,\phi'\right]$, contradicting the result in step 1. Suppose then that $\Gamma^a_\phi\left(\phi_0\right)=\Gamma^b_\phi\left(\phi_0\right)=\gamma_0$. Note that the (same) boundary conditions of the BVPs under consideration imply $\Gamma^i\left(\phi_0\right)=s_0$, $x^i\left(\phi_0\right)=1$. In turn, these observations and equations (20a)-(20b) imply $x^i_\phi\left(\phi\right)=\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}$ and $\frac{z^i_\phi(\phi_0)}{z^i(\phi_0)}=-\frac{\partial\ln A(s_0,\phi_0)}{\partial\phi}$. Log-differentiating both sides of equation (20c) and evaluating at ϕ_0 yields

$$\frac{\Gamma^i_{\phi\phi}(\phi_0)}{\Gamma^i_\phi(\phi_0)}=\frac{x^i_\phi(\phi_0)}{x^i(\phi_0)}+\frac{F_y\left(K_0^ix^i(\phi_0)\right)K_1^iK_0^ix^i_\phi(\phi_0)}{\left[1+F\left(K_0^ix^i(\phi_0)\right)K_1^i\right]}+\frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)}+\frac{g_\phi(\phi_0)}{g(\phi_0)}-[\frac{\partial\ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial s}\Gamma^i_\phi\left(\phi_0\right)+\frac{\partial\ln A\left(\Gamma^i(\phi_0),\phi_0\right)}{\partial\phi}+\frac{V_s\left(\Gamma^i(\phi_0)\right)}{V\left(\Gamma^i(\phi_0)\right)}\Gamma^i_\phi\left(\phi_0\right)+\frac{z^i_\phi(\phi_0)}{z^i(\phi_0)}],$$

$$\frac{\Gamma^i_{\phi\phi}(\phi)}{\gamma_0}=\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}+\frac{F_y\left(K_0^i\right)K_1^iK_0^i\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}}{\left[1+F\left(K_0^i\right)K_1^i\right]}+\frac{\alpha_\phi(\phi_0)}{\alpha(\phi_0)}+\frac{g_\phi(\phi_0)}{g(\phi_0)}-[\frac{\partial\ln A(s_0,\phi_0)}{\partial s}\gamma_0+\frac{\partial\ln A(s_0,\phi_0)}{\partial\phi}+\frac{V_s(s_0)}{V(s_0)}\gamma_0-\frac{\partial\ln A(s_0,\phi_0)}{\partial\phi}],$$

so,

$$\Gamma^a_{\phi\phi}\left(\phi_0\right)-\Gamma^b_{\phi\phi}\left(\phi_0\right)=K_0^b\frac{(\sigma-1)\partial\ln A(s_0,\phi_0)}{\partial\phi}\gamma_0\left\{\frac{F_y\left(K_0^b\lambda\right)K_1^b\lambda^2}{\left[1+F\left(K_0^b\lambda\right)\lambda K_1^b\right]}-\frac{F_y\left(K_0^b\right)K_1^b}{\left[1+F\left(K_0^b\right)K_1^b\right]}\right\}>0,$$

where the inequality follows from $\eta^1\left(K_0^b,\lambda\right)>\eta^1\left(K_0^b,1\right)$. The last expression implies that there is some $\phi'\in\left(\phi_0,\phi_1\right]$ such that $\Gamma^a_\phi\left(\phi\right)>\Gamma^b_\phi\left(\phi\right)$ on $\left(\phi_0,\phi'\right]$, which yields a contradiction of step 1. Accordingly, we must have $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$.

Finally, $\Gamma^a_\phi(\phi_0) < \Gamma^b_\phi(\phi_0)$ implies $\Gamma^a(\phi) < \Gamma^b(\phi)$ on some (small enough) interval (ϕ_0, ϕ''), so ϕ^+ described in the statement of the step the first time Γ^a and Γ^b intersect to the right of ϕ''.

STEP 3: *Under the assumptions of the lemma,* $\Gamma^a(\phi) < \Gamma^b(\phi)$ *on* (ϕ_0, ϕ_1).

I show that $\phi_+ = \phi_1$, where ϕ_+ was defined in step 2. Suppose for a moment that $\phi_+ < \phi_1$. If we define on $[\phi_+, \phi_1]$, $w^i(\phi) \equiv x^i(\phi)/x^i(\phi_+)$ and $y^i(\phi) = z^i(\phi)/x^i(\phi_+)$, then it is readily seen that $\{y^i, w^i(\phi), \Gamma^i\}$ solve BVP (20) in said interval, with $\alpha^i(\phi) = \alpha(\phi)$ and parameter $\overline{K}^i_0 = K^i_0 x^i(\phi_+)$. That is, $\overline{K}^a_0 = \lambda_1 \overline{K}^b_0$, where $\lambda_1 \equiv \frac{\lambda x^a(\phi_+)}{x^b(\phi_+)}$. Note that $\lambda_1 > 1$, as the BVPs associated with Γ^i satisfy the conditions of lemma 4.iv on $[\phi_0, \phi_+]$. In addition, step 2 implies $x^a(\phi_+) < x^b(\phi_+)$, so $\lambda_1 < \lambda$.

The previous discussion implies that the BVPs that $\{y^i, w^i(\phi), \Gamma^i\}$ for $i = a, b$ solve on $[\phi_+, \phi_1]$ differ only in the parameters $\{\overline{K}^i_0, K^i_1\}$, with $\overline{K}^a_0 = \lambda_1 \overline{K}^b_0$ and $K^a_1 = \lambda K^b_1$. To understand the implication of this difference, it is convenient to consider a third BVP on $[\phi_+, \phi_1]$ differing from the previous two only in the parameters $\{\overline{K}^c_0, K^c_1\}$, with $\overline{K}^c_0 = \overline{K}^a_0 = \lambda_1 \overline{K}^b_0$ and $K^c_1 = \lambda_1 K^b_1$. Given these definitions, note that the BVPs associated with $\{y^b, w^b(\phi), \Gamma^b\}$ and $\{y^c, w^c(\phi), \Gamma^c\}$ satisfy the conditions in lemma 4.vii, so step 2 above implies $\Gamma^c_\phi(\phi_+) < \Gamma^b_\phi(\phi_+)$. In addition, the BVPs associated to $\{y^a, w^a(\phi), \Gamma^a\}$ and $\{y^c, w^c(\phi), \Gamma^c\}$ satisfy the assumptions of 4.ii with $K^a_1 > K^c_1$, so $\Gamma^a_\phi(\phi_+) < \Gamma^c_\phi(\phi_+)$. These inequalities yield $\Gamma^a_\phi(\phi_+) < \Gamma^b_\phi(\phi_+)$. However, step 2 implies that $\Gamma^a_\phi(\phi_+) \geq \Gamma^b_\phi(\phi_+)$, which is a contradiction. Then it must be the case that $\phi_+ = \phi_1$.

This concludes the proof of lemma 4. ■

B.4 Section 5

B.4.1 Proof of Proposition 2

Let us start with the proof of $\phi^*_a < \phi^*_\tau$. For any $\phi^* \in [\underline{\phi}, \overline{\phi}]$, let $\{\overline{p}(.;\phi^*), \overline{r}^d(.;\phi^*), \overline{H}(.;\phi^*)\}$ denote the solution to the BVP of the open economy described in lemma 3.iii, and let $\{p(.;\phi^*), r^d(.;\phi^*), H(.;\phi^*)\}$ be the solution to the BVP of the closed economy described in lemma 1.ii where the notation emphasizes the dependence of the solution on ϕ^*. Note that this notation implies $\{p^a, r^{d,a}, H^a\} = \{p(.;\phi^*_a), r^d(.;\phi^*_a), H(;\phi^*_a)\}$ and $\{p^\tau, r^{d,\tau}, H^\tau\} = \{\overline{p}(.;\phi^*_\tau), \overline{r}^d(.;\phi^*_\tau), \overline{H}(;\phi^*_\tau)\}$, where the superscripts a and τ denote, respectively, the variables corresponding the autarky and trade equilibria of the economy under consideration. Per the discussion leading to proposition 1, the left-hand side of equation (19), which pins down the activity cutoff in the open economy, is strictly decreasing in the value of the parameter ϕ^*. Then, the result is proved if we show that the left-hand side of (19) is strictly greater than the right-hand side at $\phi^* = \phi^*_a$—i.e., if we show $\beta(\overline{r}^d(.;\phi^*_a), \phi^*_a) > \beta^a(r^d(.;\phi^*_a), \phi^*_a) = L$.[57]

First, I show that lemma 4.i implies that when the BVPs of the open and closed economy share the same boundary conditions, then the inverse matching function (matching function) corresponding to the former lies completely below (above) that of the latter. In particular, for any $\phi^* \in [\underline{\phi}, \overline{\phi}]$, $\overline{H}(\phi;\phi^*) < H(\phi;\phi^*)$ for all $\phi \in (\phi^*, \overline{\phi})$. Define $x(\phi;\phi^*) \equiv r^d(\phi;\phi^*)/\sigma f$ and $z(\phi;\phi^*) \equiv \frac{p(\phi;\phi^*)}{\sigma f}\left[L - f[1 - G(\phi^*)]\overline{M}\right]$. Then,

[57] The functions $\beta^a(.,.)$ and $\beta(.,.)$ are defined in proposition 1.

$\{z(.;\phi^*),x(.;\phi^*),H(.;\phi^*)\}$ is the unique solution to BVP (20) with parameters $K_1=0$, $\alpha(\phi;\phi^*)=1$ and boundary conditions $x(\phi^*)=1$, $H(\phi^*)=\underline{s}$ and $H(\overline{\phi})=\overline{s}$. Similarly, if we define $\overline{x}(\phi;\phi^*)\equiv\overline{r}^d(\phi;\phi^*)/\sigma f$ and $\overline{z}(\phi;\phi^*)\equiv\frac{\overline{p}(\phi)}{\sigma f}[L-fM-\int_{\phi^*}^{\overline{\phi}}nf_x\int_0^{f\overline{x}(\phi';\phi^*)\tau^{1-\sigma}/f_x}ydF(y)\,g(\phi')\,\overline{M}d\phi']$, then we can think of the solution to the open economy BVP $\{\overline{z}(.;\phi^*),\overline{x}(.;\phi^*),\overline{H}(.;\phi^*)\}$ as the unique solution to BVP (20) with parameters $\overline{K}_1=0$, $\overline{\alpha}(\phi;\phi^*)=\left[1+F\left(\frac{f\tau^{1-\sigma}}{f_x}\overline{x}(\phi;\phi^*)\right)n\tau^{1-\sigma}\right]$ and boundary conditions $\overline{x}(\phi^*)=1$, $\overline{H}(\phi^*)=\underline{s}$ and $\overline{H}(\overline{\phi})=\overline{s}$.[58] Given these definitions, it is readily seen that $\{z(.;\phi^*),x(.;\phi^*),H(.;\phi^*)\}$ and $\{\overline{z}(.;\phi^*),\overline{x}(.;\phi^*),\overline{H}(.;\phi^*)\}$ satisfy the conditions of lemma 4.i, with $\{\overline{\alpha},1\}$ taking the roles of $\{\alpha^a,\alpha^b\}$, respectively. Then, $\overline{H}(\phi;\phi^*)<H(\phi;\phi^*)$ for all $\phi\in(\phi^*,\overline{\phi})$.

I now show $\beta(\overline{r}(.;\phi^*_a),\phi^*_a)>\beta^a(r^d(.;\phi^*_a),\phi^*_a)=L$. The result in the last paragraph implies that $\{z(.;\phi^*),x(.;\phi^*),H(.;\phi^*)\}$ and $\{\overline{z}(.;\phi^*),\overline{x}(.;\phi^*),\overline{H}(.;\phi^*)\}$ satisfy the conditions of lemma 4.iii, so

$\int_{\phi^*}^{\overline{\phi}}\overline{x}(\phi;\phi^*)\frac{\overline{\alpha}(\phi;\phi^*)}{\overline{\alpha}(\phi^*;\phi^*)}g(\phi)\,d\phi>\int_{\phi^*}^{\overline{\phi}}x(\phi;\phi^*)g(\phi)\,d\phi.$

An implication of this result and $\overline{\alpha}(\phi^*;\phi^*)\geq 1$ is that total wages paid to production workers are higher in the open economy if it shares the activity cutoff with the closed economy,

$$\frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}}\overline{r}^d(\phi;\phi^*)\,\overline{\alpha}(\phi;\phi^*)\,g(\phi)\,d\phi\overline{M}=(\sigma-1)\,f\overline{\alpha}(\phi^*;\phi^*)\int_{\phi^*}^{\overline{\phi}}\overline{x}(\phi;\phi^*)\frac{\overline{\alpha}(\phi;\phi^*)}{\overline{\alpha}(\phi^*;\phi^*)}g(\phi)\,d\phi\overline{M}>\cdots$$
$$\cdots(\sigma-1)\,f\int_{\phi^*}^{\overline{\phi}}x(\phi;\phi^*)g(\phi)\,d\phi\overline{M}=\frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}}r^d(\phi;\phi^*)g(\phi)\,d\phi\overline{M}$$

In addition, per definition we have,

$$\beta^a\left(r^d(\phi;\phi^*),\phi^*\right)=\frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}}r^d(\phi;\phi^*)g(\phi)\,d\phi\overline{M}+f\left[1-G(\phi^*)\right]\overline{M}$$

$$\beta\left(\overline{r}^d(\phi;\phi^*),\phi^*\right)=\left\{\begin{array}{c}\frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}}\overline{r}^d(\phi;\phi^*)\,\overline{\alpha}(\phi;\phi^*)\,g(\phi)\,d\phi\overline{M}+\cdots\\ f\left[1-G(\phi^*)\right]\overline{M}+\int_{\phi^*}^{\overline{\phi}}nf_x\int_0^{\frac{\overline{r}^d(\phi;\phi^*)\tau^{1-\sigma}}{\sigma f_x}}ydF(y)\,g(\phi')\,\overline{M}d\phi',\end{array}\right.$$

For $\phi^*=\phi^*_a$, these observations imply

$$\beta\left(\overline{r}^d(.;\phi^*_a),\phi^*_a\right)>\beta^a\left(r^d(\phi;\phi^*_a),\phi^*_a\right)=L,$$

which is the desired result.[59]

Let us now prove the other results in the proposition, that is, $N^\tau(s)>N^a(s)$ for all $s\in[\underline{s},\overline{s})$ and proposition 2.ii. Let $N(s;\phi^*)$ be the inverse function of $H(\phi;\phi^*)$. Following the discussion above, these results can be easily proved by decomposing the total effect on the matching function into that of the increase in the exit cutoff (intensive-margin channel) and that of having an increasing share of exporters at each productivity level in the open economy (extensive-margin channel). Starting with the former, the no-crossing result in lemma 2.i and $\phi^*_a<\phi^*_\tau$ imply $N^a(s)=N(s;\phi^*_a)<N(s;\phi^*_\tau)$ on $[\underline{s},\overline{s})$.[60] Bringing

[58] Note that we are considering $\{\overline{z}(.;\phi^*),\overline{x}(.;\phi^*),\overline{H}(.;\phi^*)\}$ as the solution to a different parametrization of the BVP (20) than the one considered in section 4.2.

[59] In this derivation we used $\sigma fx(\phi;\phi^*_a)=r^{d,a}(\phi)$ and the fact that equation (14) holds in autarky.

[60] As $N(.;\phi^*)$ solves the BVP of the closed economy with activity cutoff ϕ^*, note that $N(s;\phi^*_\tau)$ is the matching function of the ancillary autarkic economy described in the paper.

the effects of exporters into the picture, lemma 4.i implies that $H(\phi;\phi_\tau^*) > \overline{H}(\phi;\phi_\tau^*) = H^\tau(\phi)$ on $(\phi_\tau^*,\overline{\phi})$, or $N(s;\phi_\tau^*) < \overline{N}(s;\phi_\tau^*) = N^\tau(s)$ on $(\underline{s},\overline{s})$. Combining these observations yield the desired result.

B.4.2 Proof of Proposition 3

Proposition 3.i

Let us start with the proof of $\phi_h^* < \phi_l^*$. For any $\phi^* \in [\underline{\phi},\overline{\phi}]$ and $i = l, h$, let $\left\{\overline{p}^i(.;\phi^*), \overline{r}^{d,i}(.;\phi^*), \overline{H}^i(;\phi^*)\right\}$ denote the solution to the BVP of the open economy described in lemma 3.iii with variable trade costs τ_i and productivity exit cutoff ϕ^* (the notation emphasizes the dependence of the solution on τ_i and ϕ^*). With this notation we have $\left\{p^i, r^{d,i}, H^i\right\} = \left\{\overline{p}^i(.;\phi_i^*), \overline{r}^{d,i}(.;\phi_i^*), \overline{H}^i(;\phi_i^*)\right\}$, where $\left\{p^i, r^{d,i}, H^i\right\}$ are the equilibrium price, revenue and inverse-matching functions of an open economy with variable trade costs τ_i. Let $\beta^i(r^d,\phi^*)$ be the function defined by the left-hand side of equation (19) in terms of r^d and ϕ^* when variable trade costs are given by τ_i.[61] Per the discussion leading to proposition 1, $\beta^i\left(\overline{r}^{d,i}(.;\phi^*),\phi^*\right)$ is strictly decreasing in the value of the parameter ϕ^*. Then, to prove the result it is enough to show $\beta^l\left(\overline{r}^{d,l}(.;\phi_h^*),\phi_h^*\right) > \beta^h\left(\overline{r}^{d,h}(.;\phi_h^*),\phi_h^*\right) = L$.

As a first step, I show that for any $\phi^* \in [\underline{\phi},\overline{\phi})$, $\overline{r}^{d,l}(\phi;\phi^*)\tau_l^{1-\sigma} > \overline{r}^{d,h}(\phi;\phi^*)\tau_h^{1-\sigma}$ for all $\phi \in [\phi^*,\overline{\phi}]$. Letting

$$\overline{x}^i(\phi;\phi^*) \equiv \overline{r}^{d,i}(\phi;\phi^*)/\sigma f,$$
$$\overline{z}^i(\phi;\phi^*) \equiv \frac{\overline{p}^i(\phi)}{\sigma f}[L - fM - \int_{\phi^*}^{\overline{\phi}} mf_x \int_0^{f\overline{x}^i(\phi';\phi^*)\tau_i^{1-\sigma}/f_x} ydF(y)\, g(\phi')\,\overline{M}d\phi'],$$

then $\{z^i(.;\phi^*), \overline{x}^i(.;\phi^*), \overline{H}^i(.;\phi^*)\}$ is the unique solution to BVP (20) with parameters $K_0^i = \frac{f}{fx}\tau_i^{1-\sigma}$, $K_1^i = n\tau_i^{1-\sigma}$, $\alpha^i(\phi;\phi^*) = 1$ and boundary conditions $\overline{x}^i(\phi^*) = 1$, $\overline{H}^i(\phi^*) = \underline{s}$ and $\overline{H}^i(\overline{\phi}) = \overline{s}$. Noting that $K_0^l = \lambda K_0^h$ and $K_1^l = \lambda K_1^h$ with $\lambda = (\tau_l/\tau_h)^{1-\sigma} > 1$, it is readily seen that $\left\{\overline{z}^i(.;\phi^*), \overline{x}^i(.;\phi^*), \overline{H}^i(.;\phi^*)\right\}$ for $i = l, k$, satisfy the conditions of lemma 4.iv, so $\overline{r}^{d,l}(\phi;\phi^*)\tau_l^{1-\sigma} > \overline{r}^{d,h}(\phi;\phi^*)\tau_h^{1-\sigma}$ for all $\phi \in [\phi^*,\overline{\phi}]$.

Let us now show $\beta^l\left(\overline{r}^{d,l}(.;\phi_h^*),\phi_h^*\right) > \beta^h\left(\overline{r}^{d,h}(.;\phi_h^*),\phi_h^*\right) = L$. To economize on space, I define the following notation

$$\begin{aligned}
\delta^i(\phi) &\equiv \left[1 + F\left(K_0^i\overline{x}^i(\phi;\phi^*)\right)K_1^i\right] \\
R^i(\phi^*) &\equiv \int_{\phi^*}^{\overline{\phi}} \overline{r}^{d,i}(\phi;\phi^*)\left[1 + F\left(\frac{\overline{r}^{d,i}(\phi;\phi^*)}{\sigma f_x}\tau_i^{1-\sigma}\right)n\tau_i^{1-\sigma}\right] g(\phi)\,d\phi\overline{M}, \\
FF^d(\phi^*) &\equiv f\left[1 - G(\phi^*)\right]\overline{M}, \text{ and } FF^{x,i}(\phi^*) \equiv \int_{\phi^*}^{\overline{\phi}} nf_x \int_0^{\frac{\overline{r}^{d,i}(\phi;\phi^*)\tau^{1-\sigma}}{\sigma f_x}} ydF(y)\, g(\phi')\,\overline{M}d\phi',
\end{aligned}$$

where $\{K_0^i, K_1^i, \overline{x}^i\}$ were defined above. These definitions and the result in the previous paragraph imply $\delta^l(\phi) > \delta^h(\phi)$ for all $\phi \in [\phi^*,\overline{\phi}]$—i.e., $\{\overline{z}^i(.;\phi^*), \overline{x}^i(.;\phi^*), \overline{H}^i(.;\phi^*)\}$ satisfy the conditions of lemma 4.v, so $R^l(\phi^*) > R^h(\phi^*)$. In addition, the result in the last paragraph also implies $FF^{x,l}(\phi^*) > FF^{x,h}(\phi^*)$.

[61] Note that $\beta^i(.,.)$ is just the function $\beta(.,.)$ defined in proposition 1, where the superscript i in the current notation emphasizes the dependence of this function on τ_i.

These inequalities and the definition of β^i yield

$$\begin{aligned}\beta^l\left(\overline{r}^{d,l}\left(\phi;\phi_h^*\right),\phi_h^*\right) &= \tfrac{\sigma-1}{\sigma}R^l\left(\phi_h^*\right)+FF^d(\phi_h^*)+FF^{x,l}\left(\phi_h^*\right)\\ &> \tfrac{\sigma-1}{\sigma}R^h\left(\phi_h^*\right)+FF^d(\phi_h^*)+FF^{x,h}\left(\phi_h^*\right)\\ &= \beta^h\left(\overline{r}^{d,h}\left(\phi;\phi_h^*\right),\phi_h^*\right)=L.\end{aligned}$$

As discussed above, this result implies $\phi_h^* < \phi_l^*$.

Finally, the continuity of the matching functions and $\phi_h^* < \phi_l^*$ imply that there is a skill level $s' \in (\underline{s}, \overline{s}]$ such that $N^l(s) > N^h(s)$ on $[\underline{s}, s')$—i.e., inequality necessarily increases among the least-skilled workers in the economy after a trade liberalization.

Proposition 3.ii

As discussed in the text, the distributional effects of the extensive-margin channel are theoretically ambiguous. Accordingly, below I formally derive the impact on relative wages of the other two channels, the selection-into-activity and the intensive-margin channels. I start by defining some notation. In the sequel, $\{z(\phi;\phi^*,\alpha),\ x(\phi;\phi^*,\alpha),\ H(\phi;\phi^*,\alpha)\}$ denotes the unique solution to BVP (20) with constant $K_1 = 0$, parameter function α, and boundary conditions $\{\overline{x}(\phi^*) = 1, H(\phi^*) = \underline{s}, H(\overline{\phi}) = \overline{s}\}$, where the notation emphasizes the dependence of the solution on $\{\phi^*, \alpha\}$. In addition, I will use $N(\phi;\phi^*,\alpha)$ to denote the inverse of $H(\phi;\phi^*,\alpha)$. For $i = l, h$, let $\left\{\phi_i^*, p^i, r^{d,i}, H^i\right\}$ be the activity cutoff, price, domestic revenue and inverse-matching functions of the two open economies in the statement of the proposition (these economies differ only in the variable trade costs they face, with $\tau_l < \tau_h$). Defining the parameter functions $\alpha^i(\phi) \equiv \left[1 + F\left(\frac{\tau_i^{1-\sigma}}{\sigma f_x} r^{d,i}(\phi)\right) n\tau_i^{1-\sigma}\right]$ for $i = l, h$, we can think of the BVPs associated with each H^i as particular parameterizations of BVP (20) with $K_1 = 0$ and $\alpha = \alpha^i$.[62] In the notation defined here,

$$\begin{aligned}x\left(\phi;\phi_i^*,\alpha^i\right) &= r^{d,i}\left(\phi;\phi_i^*\right)/\sigma f\\ z\left(\phi;\phi_i^*,\alpha^i\right) &= \frac{p^i(\phi)}{\sigma f}[L - fM - \int_{\phi^*}^{\overline{\phi}} nf_x \int_0^{\frac{r^{d,i}(\phi;\phi_i^*)\tau^{1-\sigma}}{\sigma f_x}} ydF(y)\, g(\phi')\, \overline{M} d\phi']\\ H\left(\phi;\phi_i^*,\alpha^i\right) &= H^i\end{aligned}$$

After these preliminaries we are ready to prove the claim.

Let us start with the **selection-into-activity channel**. As discussed in the text, the matching functions N_0 and N^h in figure 2 differ only in their activity cutoffs—i.e., $N_0 = N\left(\phi;\phi_l^*,\alpha^h\right)$ and $N^h = N\left(\phi;\phi_h^*,\alpha^h\right)$. Accordingly, the no-crossing result in lemma 2.i implies $N_0(s) > N^h(s)$ on $[\underline{s}, \overline{s})$. Note that by sharing the same parameter function α^h, the economies associated with N_0 and N^h have the same fraction of exporters at each productivity (among active firms) and face the same variable costs. Accordingly, their difference captures the effects of the selection-into-activity channel on relative wages.

[62] See the proof of proposition 2.

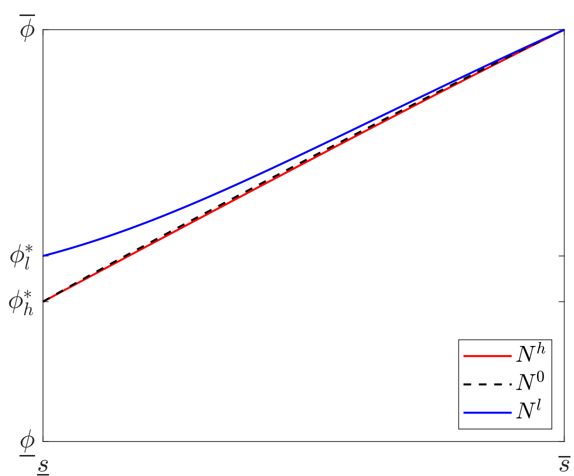
Let us now turn to the **intensive-margin channel**. Define $\alpha^1(\phi) \equiv \left[1 + F\left(\frac{\tau_h^{1-\sigma}}{\sigma f_x} r^{d,h}(\phi)\right) n\tau_l^{1-\sigma}\right]$ and note that $\alpha^1(\phi)$ differs from $\alpha^h(\phi)$ only in the value of the variable trade cost outside the function F. In addition, note that for any pair $\phi'', \phi' \in [\phi^*, \overline{\phi}]$ such that $\phi'' > \phi'$ and $F\left(\tau_h^{1-\sigma} r^{d,h}(\phi'')/\sigma f_x\right) > 0$, we have $\alpha^1(\phi'')/\alpha^1(\phi') > \alpha^h(\phi'')/\alpha^h(\phi')$. As discussed in the text, the matching functions N_0 and N_1 in figure 2 differ only in their parameter function α—i.e., $N_0 = N\left(\phi; \phi_l^*, \alpha^h\right)$ and $N_1 = N\left(\phi; \phi_l^*, \alpha^1\right)$. Accordingly, the BVPs associated with N_0 and N_1 satisfy the conditions of lemma 4.i, so $N_1(s) > N_0(s)$ on $[\underline{s}, \overline{s})$.

Proposition 3.iii

To prove the result, it is convenient to break the changes in the BVP of the open economy introduced by the liberalization in two parts, the change associated to the decline in variable trade costs and the change associated to the rise in the activity cutoff (allowing the set of exporters to adjust in each case). Starting with the former, let N_0 be the matching function resulting from reducing τ_h to τ_l in the BVP of the open economy before the liberalization, keeping the activity cutoff unchanged. If the assumption on η_1^F is satisfied, then it is readily seen that F and the open-economy BVPs associated with N^h and N^0 satisfy the conditions in lemma 4.vii with $K_0^h = f\tau_h^{1-\sigma}/f_x$, $K_1^h = n\tau_h^{1-\sigma}$, $K_i^0 = \lambda K_i^h$, and $\lambda = (\tau_l/\tau_h)^{1-\sigma} > 1$. Accordingly, $N^0(s) > N^h(s)$ on $(\underline{s}, \overline{s})$ as shown in figure 11.

Figure 11: Trade Liberalization under Sufficient Conditions

Proof.



■

Note: The solid red and blue lines represent, respectively, the pre- (N^h) and post-liberalization (N^l) matching functions. The dashed black line (N^0) represents the solution to the BVP of the open economy with $\tau = \tau_l$ and $\phi^* = \phi_h^*$. When η_1^F satisfies the sufficient conditions in proposition 3.iii, N^0 lies above N^h. When η_0^F satisfies the sufficient conditions in said proposition, N^l lies above N^0.

Now consider the change in the matching function associated with the rise in the activity cutoff—i.e., the difference between N_0 and N^l in figure 11. Suppose that N^0 and N^l intersect on $(\underline{s}, \overline{s})$ with the first intersection occurring at s', namely $N^0(s') = N^l(s') = \phi'$. If for $i = 0, l$, we define on $\left[\phi', \overline{\phi}\right]$

the functions $w^i(\phi) \equiv \frac{r^{d,i}(\phi)}{r^{d,i}(\phi')}$ and $y^i(\phi) \equiv \frac{p^i(\phi)}{r^{d,i}(\phi')\overline{M}}[L - fM^i - \int_{\phi_i^*}^{\overline{\phi}} nf_x \int_{\underline{y}}^{\frac{r^{d,i}(\phi)\tau_l^{1-\sigma}}{\sigma f_x}} ydF(y)\, g(\phi)\, \overline{M} d\phi]$, then $\{w^i, y^i, H^i\}$ is the unique solution to BVP (20) with parameters $\alpha^i(\phi) = 1$, $K_0^i = \frac{r^{d,i}(\phi')\tau_l^{1-\sigma}}{\sigma f}$, $K_1^i = n\tau_l^{1-\sigma}$ and boundary conditions $w^i(\phi') = 1$, $H^i(\phi') = s'$ and $H^i(\overline{\phi}) = \overline{s}$. In addition, note that the log-supermodularity of A and $H^l(\phi) < H^0(\phi)$ on $[\phi_l^*, \phi')$ implies $r^{d,0}(\phi') > r^{d,l}(\phi')$, so $K_0^0 > K_0^l$. Accordingly, if the assumption on η_0^F is satisfied, then its is readily seen that F and the open-economy BVPs associated with N^l and N^0 satisfy the conditions of lemma 4.vi on $[\phi', \overline{\phi}]$, so $H_\phi^l(\phi') < H_\phi^0(\phi')$. However, $H^l(\phi) < H^0(\phi)$ on $[\phi_l^*, \phi')$ implies $H_\phi^l(\phi') \geq H_\phi^0(\phi')$, which is a contradiction. Then it must be the case that N^l and N^0 do not intersect on $(\underline{s}, \overline{s})$, so N^l lies strictly above N^0 on $[\underline{s}, \overline{s})$ as shown in the picture.

Combining the last two results we get $N^l(s) > N^h(s)$ on $[\underline{s}, \overline{s})$, so inequality is pervasively higher after the liberalization. This concludes the proof of the proposition.

B.5 Section 6

B.5.1 Free-Entry Equilibrium in the Closed Economy

In the free-entry model, the mass of firms in the industry, $\overline{M}$, is an additional endogenous variable. As described in the main text, there is an unbounded pool of prospective firms that can enter the industry by incurring a fixed entry-cost of $f^e V(s)$ units of each skill $s \in S$. Upon entry, firms obtain their productivity as independent draws from the distribution G, as explained in section 2.2. Note that the new free-entry assumption does not affect the basic structure of the model described in section 2, so equations (1)-(5) continue to hold.

The analysis of the closed-economy equilibrium in section 3 is valid for any mass of firms, $\overline{M}$, so it applies almost unchanged to the free-entry model once $\overline{M}$ has been determined. In fact, conditional on $\overline{M}$, the analysis needs to be modified only to account for the presence of fixed entry-costs—i.e., L must be replaced with $L - f^e\overline{M}$ throughout the analysis. A *free-entry* condition provides the additional equilibrium condition to pin down the mass of firms.

In the free-entry model, the labor market clearing condition is given by

$$LV(s) = \int_{\phi^*}^{\overline{\phi}} l^d(s, \phi) \frac{g(\phi)}{[1 - G(\phi^*)]} d\phi M + MfV(s) + \overline{M}f^e \text{ for all } s \in S. \tag{79}$$

With unrestricted entry, prospective entrants must be indifferent between entering and not entering the industry—i.e., expected profits from entering must equal the cost of entry, $[1 - G(\phi^*)]\, \overline{\pi}^d = f^e$, where $\overline{\pi}^d$ is the average domestic profit among active firms. Per the optimal pricing rule, this *free-entry* condition can be written as follows,

$$\int_{\phi^*}^{\overline{\phi}} \left[\frac{r^d(\phi)}{\sigma} - f\right] g(\phi)\, d\phi = f^e. \tag{80}$$

Definition 3 *A free-entry equilibrium of the closed economy is a mass of firms $\overline{M} > 0$, a mass of active firms $M > 0$, a productivity activity-cutoff , $\phi^* \in (\underline{\phi}, \overline{\phi})$, an output function $q^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a labor allocation function $l^d : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, a price function $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and a wage schedule $w : S \to \mathbb{R}_+$ such that the following conditions hold,*
(i) consumers behave optimally, equations (1) and (2);
(ii) firms behave optimally given their technology, equations (3), (5), (7), and (8);
(iii) goods and labor markets clear, equations (6) and (79), respectively;
(iv) the numeraire assumption holds, $\overline{w} = 1$;
(v) the free-entry condition holds, equation (80).

Given the equilibrium activity cutoff, ϕ^*, the price, domestic revenue, and inverse-matching functions, $\{p, r^d, H\}$, solve a BVP that is almost identical to the one defined in lemma 1.ii for the no-free-entry model. The only difference lies in the slope of the inverse-matching function, which is now given by

$$H_\phi(\phi) = \frac{r^d(\phi)\, g(\phi)\, \overline{M}}{A(H(\phi), \phi)\left[L - fM - f^e \overline{M}\right] V(H(\phi))\, p(\phi)}. \tag{81}$$

The discussion in section 4.2 implies that, for a given activity cutoff ϕ^*, the functions r^d and H that solve this BVP do not depend on the mass of firms nor the mass of production workers. Noting that equations (13) and (81) may differ only in these parameters, the last observation implies that, for a given ϕ^*, the closed-economy BVPs of the no-free-entry and free-entry models share the same solution functions r^d and H.[63]

As the revenue function r^d depends only on ϕ^*, the free-entry condition (80) can be used to determine the equilibrium activity cutoff, ϕ^*. Finally, combining the equilibrium relationship $L = \overline{M} f^e + Mf + \frac{\sigma-1}{\sigma} M \overline{r}^d$ (the counterpart of condition (14) in the no-free-entry model) and the free-entry condition, we can express the mass of firms as a function of exogenous variables and the activity cutoff ϕ^*,

$$\overline{M} = \frac{L}{\sigma f^e + \sigma f\left[1 - G(\phi^*)\right]}. \tag{82}$$

I summarize this discussion in the following lemma.

Lemma 5 *In a free-entry equilibrium of the closed economy with activity cutoff $\phi^* \in (\underline{\phi}, \overline{\phi})$ the following conditions hold.*
(i) There exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$, (with inverse function H) such that (i) $l^d(s, \phi) > 0$ if and only if $N(s) = \phi$, (ii) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$.
(ii) The wage schedule w is continuously differentiable and satisfies (10).
(iii) The price, revenue, and matching functions, $\{p, r^d, N(H)\}$, are continuously differentiable. Given ϕ^*, the triplet $\{p, r^d, H\}$ solves the BVP comprising the differential equations {(11), (12), (81)} and the boundary conditions $r^d(\phi^*) = \sigma f$, $H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$.

[63] These two BVPs are equivalent to the same parametrization of BVP (20).

(iv) The activity cutoff ϕ^* and the revenue function r^d satisfy the free-entry condition (80).
(v) The mass of firms in the industry, $\overline{M}$, is given by (82).
Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$, and functions $p, r^d : [\phi^*, \overline{\phi}] \rightarrow \mathbb{R}_+$ and $H : [\phi^*, \overline{\phi}] \rightarrow S$ satisfy conditions (ii)-(iv), then they are, respectively, the productivity activity-cutoff, the price function, the revenue function, and the inverse of the matching function of a free-entry equilibrium of the closed economy.

The discussion preceding proposition 1 implies that $r^d(\phi)$ decreases with ϕ^*, making the left-hand side of (80) strictly decreasing in ϕ^*. If the fixed entry costs are not too high, then there is a unique activity cutoff ϕ^* that solves (80). In turn, this result implies that there is a unique free-entry equilibrium of the open economy.

B.5.2 Free-Entry Equilibrium in the Open Economy

The similarities between the analyses of the closed-economy equilibrium in the no-free-entry and free-entry models extend to the open economy. In particular, replacing L with $L - f^e\overline{M}$ throughout the analysis in section 4 yields the characterization of the open-economy equilibrium in the free-entry model, conditional on the mass of firms $\overline{M}$. The free-entry condition provides the additional equilibrium condition to determine $\overline{M}$.

The labor market clearing condition in the open economy is given by

$$LV(s) = \begin{array}{c} \int_{\phi^*}^{\overline{\phi}} [l^d(s,\phi)\, g(\phi)\, \overline{M} + l^x(s,\phi)\, M^x(\phi)] d\phi + \cdots \\ fMV(s) + nf^x \int_0^{\frac{\tau^{1-\sigma} r^d(\phi)}{\sigma f_x}} ydF(y)\, M^x(\phi)\, V(s) + f^e\overline{M}V(s) \end{array} \quad \text{for all } s \in S. \tag{83}$$

As before, unrestricted entry implies that expected profits from entering the industry must equal the cost of entry, $[1 - G(\phi^*)]\left[\overline{\pi}^d + \overline{\pi}^x\right] = f^e$, where $\overline{\pi}^d$ and $\overline{\pi}^x$ are, respectively, the average domestic and export profit among active firms.[64] Per the optimal pricing rule, this *free-entry* condition can be written as shown in equation (22) in the main text.

Definition 4 *A free-entry equilibrium of the open economy is a mass of firms $\overline{M}$, an activity cutoff ϕ^*, a mass of active firms $M > 0$, a mass of exporters $M^x(\phi) > 0$ for each productivity level $\phi \geq \phi^*$, output functions $q^d, q^x : [\phi^*, \overline{\phi}] \rightarrow \mathbb{R}_+$, labor allocations functions $l^d, l^x : S \times [\phi^*, \overline{\phi}] \rightarrow \mathbb{R}_+$, a price function $p : [\phi^*, \overline{\phi}] \rightarrow \mathbb{R}_+$ and a wage schedule $w : S \rightarrow \mathbb{R}_+$ such that the following conditions hold,*
(i) consumers behave optimally, equations (1) and (2);
(ii) firms behave optimally given their technology, equations (3), (5), (7), (8) and (16);
(iii) goods and labor markets clear, equations (6), (15) and (22);
(iv) the numeraire assumption holds, $\overline{w} = 1$;
(v) the free-entry condition holds, equation (22).

[64] Note that $\overline{\pi}^x$ is not the average export profits among exporters, but among all active firms.

Given the equilibrium activity cutoff ϕ^*, the price, domestic revenue, and inverse-matching functions, $\{p, r^d, H\}$, solve a BVP that is almost identical to the one defined in lemma 3.iii for the no-free-entry model. The only difference lies in the slope of the inverse-matching function, which is now given by

$$H_\phi(\phi) = \frac{r^d(\phi)\left[1+F\left(\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}\right)n\tau^{1-\sigma}\right]g(\phi)\overline{M}}{A(H(\phi),\phi)V(H(\phi))p(\phi)\left[L-fM-f^e\overline{M}-\int_{\phi^*}^{\overline{\phi}} nf^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF(y)g(\phi')\overline{M}d\phi'\right]}. \tag{84}$$

Noting that equations (18) and (84) may differ only in the mass of firms or the mass of production workers, the discussion in the preceding section implies that, for a given ϕ^*, the open-economy BVPs of the no-free-entry and free-entry models share the same solution functions r^d and H.

As before, the free-entry condition (22) can be used to determine the equilibrium activity cutoff. Finally, the equilibrium relationship, $L = \overline{M}f^e + Mf + \int_{\phi^*}^{\overline{\phi}} nf^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF(y)\, g(\phi')\, \overline{M}d\phi' + \frac{\sigma-1}{\sigma}M\overline{r}^d + \frac{\sigma-1}{\sigma}M\overline{r}^x$, can be combined with the free-entry condition to express the mass of firms in the industry as a function of exogenous parameters, the activity cutoff ϕ^* and the revenue function r^d,

$$\overline{M} = \frac{L}{\sigma\left[f^e + f\left[1-G(\phi^*)\right] + \int_{\phi^*}^{\overline{\phi}} nf^x \int_0^{\frac{r^d(\phi')\tau^{1-\sigma}}{\sigma f^x}} ydF(y)\, g(\phi')\, d\phi'\right]}. \tag{85}$$

I summarize this discussion in the following lemma.

Lemma 6 *In a free-entry equilibrium of the open economy with activity cutoff $\phi^* \in (\underline{\phi}, \overline{\phi})$ the following conditions hold.*

(i) There exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$, (with inverse function H) such that (i) $l^d(s,\phi) + l^x(s,\phi) > 0$ if and only if $N(s) = \phi$, (ii) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$.

(ii) The wage schedule w is continuously differentiable and satisfies (10)

(iii) The price, domestic revenue, and matching functions, $\{p, r^d, N\}$, are continuously differentiable. Given ϕ^*, the triplet $\{p, r^d, H\}$ solves the BVP comprising the system of differential equations $\{(11), (12), (84)\}$ and the boundary conditions $r^d(\phi^*) = \sigma f$, $H(\phi^*) = \underline{s}$, $H(\overline{\phi}) = \overline{s}$.

(iv) The activity cutoff ϕ^* and the revenue function r^d satisfy the free-entry condition (22).

(v) The mass of firms in the industry, $\overline{M}$, is given by (85).

Moreover, if a number $\phi^* \in (\underline{\phi}, \overline{\phi})$, and functions $p, r^d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $H : [\phi^*, \overline{\phi}] \to S$ satisfy the conditions (iii)-(iv), then they are, respectively, the activity cutoff, the price function, the domestic revenue function, and the inverse-matching function of a free-entry equilibrium of the open economy.

B.5.3 Proof of Proposition 4

In the free-entry model the activity cutoff may increase or decrease when the economy starts trading. The reasons behind this ambiguity are discussed in the text. In addition, as stated in the text, proposition 4.i considers essentially the same case as proposition 2, so the arguments in the proof of the latter also applies to the former. Here I focus on Proposition 4.ii.

Proposition 4.ii

Let $\phi_\tau^* < \phi_a^*$. If $N^\tau(s) < N^a(s)$ for all $s \in [\underline{s}, \overline{s})$, then lemma 2.ii implies that $r^{d,\tau}(\phi) > r^{d,a}(\phi)$ for all $\phi \geq \phi_a^*$, so domestic profits in the open economy are necessarily higher than in autarky. With strictly positive export profits, this observation implies that total average profits must be higher in the open economy, violating the free entry condition (22). Accordingly, $N^\tau(s)$ must lie above $N^a(s)$ for some values of s, implying that $N^\tau(s)$ and $N^a(s)$ must intersect at least once on $(\underline{s}, \overline{s})$.

Next, I show that $N^\tau(s)$ and $N^a(s)$ intersect exactly once on $(\underline{s}, \overline{s})$. The argument is more easily stated in terms of the inverse functions H^τ and H^a. Let ϕ_0 be the first time that H^τ and H^a intersect on $(\phi_a^*, \overline{\phi})$. Note that H^τ and H^a are the unique solutions to parameterizations of BVP (20) that differ only in the parameter function α^i, with $K_1^i = 0$ for $i = \tau, a$, $\alpha^\tau(\phi) = 1 + F\left(\frac{r^{d,\tau}(\phi)}{\sigma f_x}\tau^{1-\sigma}\right)$ and $\alpha^a(\phi) = 1$. Then, an immediate application of lemma 4.i yields $H^\tau(\phi) < H^a(\phi)$ on $(\phi_0, \overline{\phi})$, so H^τ and H^a ($N^\tau(s)$ and $N^a(s)$) intersect exactly once on $(\phi_a^*, \overline{\phi})$ ($(\underline{s}, \overline{s})$) at ϕ_0 ($s_0 = H^i(\phi_0)$).

The last result implies that, in the open economy, inequality is lower among workers with skill levels below s_0, but higher among workers with skill level above s_0. Put another way, opening to trade leads to wage polarization. The effects of the intensive- and extensive-margin channels can be proved by adapting the arguments in proposition 2.ii.

C Calibration

This appendix explains in more detail the calibration strategy sketched in section 7 . Section C.1 describes calibration assumptions based on parameter estimates and moment restrictions from the literature, as well as some restrictions that these assumptions impose on other elements of the model. Section C.2 assumes specific functional forms for other elements of the model that are compatible with previous assumptions. Given these assumptions and the model's equilibrium conditions, section C.3 backs out the implied expressions for all remaining endogenous and exogenous elements of the model. Using these expressions, section C.4 shows how to compute relevant empirical moments in the model. Section C.5 describes the main calibration approach. Finally, section C.6 shows that, given the calibration approach, some parameters of the model do not affect the model-implied values for targeted moments nor wage inequality in the calibrated equibrium. Accordingly, this paramters are chose as normalizations. In addition, this section also shows that the calibrated CDF of fixed export costs satisfies the sufficient conditions in proposition3.iii.

As discussed in the main text, the goal of the quantitative analysis based on this calibration is to identify the most likely broad distributional effects of higher trade openness through the labor reallocation mechanisms emphasized in the paper. In particular, the quantitative analysis sheds light into the *relative* importance of each the three channels defined in section 5—selection-into-activity, intensive-margin and extensive-margin channels. That said, the reader should exert some caution in interpreting the quantitative implications of the model regarding the *total* change in overall measures of wage inequality following a decline in trade costs. These implications critically depend on the parameter ρ, which has not been estimated in this paper.

C.1 Assumptions Based on Estimates and Moment Restrictions from the Literature

As I discussed in the main text, my calibration approach is partly based on Melitz and Redding (2015), henceforth MR. Specifically, as in MR, I set the elasticity of substitution between final goods to four, $\sigma = 4$, and make the model match the average exports-to-sales ratio among Portuguese manufacturing firms, $n\tau^{1-\sigma}/(1+n\tau^{1-\sigma}) = 0.31$, which yields a value for $n\tau^{1-\sigma}$. As I discuss later, all relevant calibrated variables—including the moments targeted in the calibration as well as wage inequality in the calibrated equilibrium—depend on $\{n, \tau\}$ only through $n\tau^{1-\sigma}$. The same is true regarding the counterfactual implications of the calibrated model discussed in next section. As such, I proceed without picking specific values for $\{n, \tau\}$, as such a choice does not affect any of the results discussed below. The reader should note, however, that the level variable trade costs determines the scope for further trade liberalization in the model, which could affect the interpretation of some counterfactual results. In sum,

$$\sigma = 4; \qquad n\tau^{1-\sigma} = 0.45. \tag{86}$$

Following MR, I make assumptions that guarantee that firm's revenue in the model, r^d, is distributed approximately Pareto with shape parameter equal to 1. A well-known mathematical result is that $r^d(\phi)$ is distributed Pareto when it is a power function of ϕ and the latter is itself distributed Pareto. Accordingly, I assume that ϕ is distributed Pareto with minimum parameter $\underline{\phi} = 1$ and shape parameter θ and that r^d is of the form $r^d(\phi) = \sigma f(\phi/\phi^*)^\beta$. As a result, r^d is distributed Pareto with minimum parameter $\underline{r}^d = \sigma f$ and shape parameter θ/β. Letting $G^r(r^d)$ be the CDF of r^d, the previous discussion can be summarized as

$$\left.\begin{array}{c} G(\phi) = 1 - \phi^{-\theta} \\ r^d(\phi) = \sigma f(\phi/\phi^*)^\beta \end{array}\right\} \Rightarrow G^r\left(r^d\right) = 1 - \left(\frac{r^d}{\sigma f}\right)^{-\theta/\beta}. \tag{87}$$

To matched the stylized fact mentioned above, I set $\beta = \theta$ in all computations.

C.1.1 Implications for Other Elements of the Model

The assumptions in (87) and the model's equilibrium conditions impose restrictions on other model's elements, as the endogenous revenue function r^d depends on the productivity function $A(s, \phi)$ and on the shape of the equilibrium matching function, which in turn depends on other primitives, including the distributions of worker skill, firm productivity, and fixed export costs. For example, these restrictions limit the set of compatible functional forms for the productivity function $A(s,\phi)$ as I discuss below.

Note that the functional form assumed for $r^d(\phi)$ in (87) and equilibrium condition (12) in the main text imply

$$\frac{r^d_\phi(\phi)}{r^d(\phi)} = \frac{\beta}{\phi} = (\sigma-1)\frac{\partial \ln A(H(\phi),\phi)}{\partial\phi}. \tag{88}$$

Now consider the following functional form for the productivity function, $A(s,\phi) = B_0^A \exp\left(B_1^A s^{\alpha_s}\phi^{\alpha_\phi}\right)$ with $B_0^A > 0$. Using the last condition to solve for H yields $H(\phi) = \left[\frac{\beta}{B_1(\sigma-1)\alpha_\phi}\right]^{\frac{1}{\alpha_s}} \phi^{-\frac{\alpha_\phi}{\alpha_s}}$. For $B_1^A > 0$, the assumptions $A_\phi, A_s > 0$ imply $\alpha_s, \alpha_\phi > 0$. In turn, this observation implies that H is decreasing in ϕ, which is a contradiction. Alternatively, if $B_1^A < 0$, then $A_\phi, A_s > 0$ implies $\alpha_s, \alpha_\phi < 0$, in turn implying that A is not log-supermodular. Accordingly, this particular functional form for $A(s,\phi)$ is incompatible with the assumptions in (87) and the model's equilibirum conditions.

C.2 Other Compatible Functional-Form Assumptions

I assume that $A(s,\phi)$ takes the following CES functional form, $A(s,\phi) = B_0^A\left[\alpha_s s^\rho + \alpha_\phi\phi^\rho\right]^{\frac{B_1^A}{\rho}}$, with $B_0^A, B_1^A, \alpha_s, \alpha_\phi > 0$ and $\alpha_s + \alpha_\phi = 1$. Under some parameter conditions, this specification of $A(s,\phi)$ is compatible with the restrictions imposed by the assumptions in (87) and the model's equilibrium conditions. Below, I discuss some of the these parameter conditions.

The CES functional form implies $\frac{\partial^2 \ln A(s,\phi)}{\partial\phi\partial s} = -\frac{\rho B_1^A \alpha_\phi\phi^{\rho-1}\alpha_s s^{\rho-1}}{\left[\alpha_s s^\rho + \alpha_\phi\phi^\rho\right]^2}$, so $A(s,\phi)$ is strict log-supermodular if and only if $\rho < 0$. In addition, combining this functional form with condition (88) yields $H(\phi) = B_0^H\phi$, with $B_0^H = \left[\frac{\left[(\sigma-1)B_1^A-\beta\right]\alpha_\phi}{\beta\alpha_s}\right]^{\frac{1}{\rho}}$. Accordingly, H is strictly increasing only if $(\sigma-1)B_1^A > \beta$. This discussion can be summarized as

$$\boxed{\begin{aligned} &A(s,\phi) = B_0^A\left[\alpha_s s^\rho + \alpha_\phi\phi^\rho\right]^{\frac{B_1^A}{\rho}}, \text{ with } B_0^A, B_1^A, \alpha_s, \alpha_\phi > 0,\ \alpha_s+\alpha_\phi = 1,\ \rho < 0 \Rightarrow \\ &\Rightarrow H(\phi) = B_0^H\phi, \text{ with } B_0^H = \left[\frac{\left[(\sigma-1)B_1^A-\beta\right]\alpha_\phi}{\beta\alpha_s}\right]^{\frac{1}{\rho}}; \qquad B_0^H > 0 \Rightarrow (\sigma-1)B_1^A > \beta \end{aligned}} \tag{89}$$

To facilitate the estimation of the model, I also assume a functional form for the endogenous fraction of firms with productivity ϕ that export in the calibrated equilibrium, $FX(\phi) \equiv F\left(r^d(\phi)\tau^{1-\sigma}/\sigma f^x\right)$.

This functional form is given by[65]

$$FX(\phi) = \begin{cases} 0 & \text{if } \phi < \phi_{lb}^{FX} \\ \frac{\left[\left(\phi_{lb}^{FX}\right)^{-\gamma} - \phi^{-\gamma}\right]}{\left[\left(\phi_{lb}^{FX}\right)^{-\gamma} - \left(\phi_{ub}^{FX}\right)^{-\gamma}\right]} & \text{if } \phi_{lb}^{FX} \leq \phi \leq \phi_{ub}^{FX} \\ 1 & \text{if } \phi > \phi_{ub}^{FX} \end{cases} \tag{90}$$

Why do I make a functional form assumption directly on the endogenous function $FX(\phi)$ instead of, say, the exogenous CDF of fixed export costs? Because this is the element of the model that matters when it comes to matching the fraction of firms that are exporters in each decile of value-added per worker in the Portuguese data. Moreover, once $FX(\phi)$ is determined and a value for f_x is chosen, F and can be recovered as a residual from its definition, $FX(\phi) \equiv F\left(r^d(\phi)\,\tau^{1-\sigma}/\sigma f^x\right)$.

C.3 Implied Expressions for Other Elements of the Model.

The restrictions imposed by the assumptions made so far on $\{G(\phi), r^d(\phi), A(s,\phi), FX(\phi)\}$—summarized by (86), (87), (89) and (90)—and the model's equilibrium conditions allow me to back out the implied functional forms of all remaining endogenous and exogenous elements of the model, including those of the exogenous distributions of worker skill and fixed export costs. I discuss these calculations below.

C.3.1 Price Function

The differential equations involving $r_\phi(\phi)$ and $p_\phi(\phi)$ in the BVP of the open (or closed) economy, equations (11) and (12), yield $r_\phi(\phi)/r(\phi) = -(\sigma-1)\,p_\phi(\phi)/p(\phi) = \beta/\phi$, so

$$p(\phi) = B_0^p\,(\phi/\phi^*)^{-\frac{\beta}{\sigma-1}}\,, \tag{91}$$

where B_0^p is a constant to be determined.

C.3.2 Distribution of Skills

Given the expressions for $r^d(\phi)$ and $g(\phi)$ implied by (87), the expressions for $A(s,\phi)$ and $H(\phi)$ in (89), and the expression for $p(\phi)$ in (91), the differential equation of the open-economy BVP involving H_ϕ, equation (18), implies

$$V(H(\phi)) = B_0^V\,\phi^{\frac{\beta\sigma}{(\sigma-1)} - B_1^A}\left[1 + FX(\phi)\,n\tau^{1-\sigma}\right] g(\phi)\,.$$

In addition, B_0^V can be determined from the condition that V must integrate to one (density function).[66]

[65] Note that the functional form assumed for $FX(\phi)$ is just the CDF of a truncated Pareto ditribution with shape parameter γ.

[66] This integral condition, $\int_{\underline{s}}^{\overline{s}} V(s)\,ds = 1$, can be expressed as $\int_{\phi^*}^{\overline{\phi}} V(H(\phi))\,H_\phi(\phi)\,d\phi = 1$ after changing variables of integration.

Indeed, the functional form assumed for $FX(\phi)$ in (90) implies that there is a closed-form expression for B_0^V, reducing the computational cost associated to numerical integration.

Before deriving an expression for B_0^V, I define some notation that will facilitate the exposition,

$$v_0(\phi) \equiv \phi^{\frac{\beta\sigma}{(\sigma-1)}-B_1^A} g(\phi); \quad v_1(\phi) \equiv \phi^{\frac{\beta}{(\sigma-1)}-B_1^A}; \quad v(\phi) \equiv v_0(\phi)\left[1 + FX(\phi)\, n\tau^{1-\sigma}\right].$$

With these definitions, the integral condition on V implies $B_0^V = 1/[B_0^H \int_{\phi^*}^{\overline{\phi}} v(\phi)\, d\phi]$. Below I derive a closed-form expression for the integral $\int_{\phi^*}^{\overline{\phi}} v(\phi)$ as a function of the limits of integration.

It is convenient to break this integral into two terms,

$$\int_{\phi^*}^{\overline{\phi}} v(\phi)\, d\phi = \int_{\phi^*}^{\overline{\phi}} v_0(\phi)\, d\phi + \int_{\phi_x^*}^{\overline{\phi}} v_0(\phi)\, FX(\phi)\, n\tau^{1-\sigma} d\phi.$$

Operating on the first term while using $\beta = \theta$ yields $\int_{\phi^*}^{\overline{\phi}} v_0(\phi)\, d\phi = C_0[v_1(\overline{\phi}) - v_1(\phi^*)]$, with $C_0 \equiv \frac{\beta}{[\underline{\phi}^{-\beta} - \overline{\phi}^{-\beta}][\frac{\beta}{(\sigma-1)} - B_1^A]}$. Note that the condition $\beta < B_1^A(\sigma-1)$ in (89) implies $\int_{\phi^*}^{\overline{\phi}} v_0(\phi)\, d\phi > 0$.

Integrating by parts on the second term of $\int_{\phi^*}^{\overline{\phi}} v(\phi)\, d\phi$ yields

$$\int_{\phi^*}^{\overline{\phi}} v_0(\phi)\, FX(\phi)\, n\tau^{1-\sigma} d\phi = n\tau^{1-\sigma}\left\{[C_0 v_1(\phi)\, FX(\phi)|_{\phi^*}^{\overline{\phi}} - \int_{\phi^*}^{\overline{\phi}} C_0 v_1(\phi)\, FX_\phi(\phi)\, d\phi\right\},$$

with the first term in the last expression given by $[C_0 v_1(\phi)\, FX(\phi)|_{\phi^*}^{\overline{\phi}} = C_0 n\tau^{1-\sigma} v_1(\overline{\phi})\, FX(\overline{\phi}) - v_1(\phi^*)\, FX(\phi^*)$ and the second term given by $\int_{\phi^*}^{\overline{\phi}} v_1(\phi)\, FX_\phi(\phi)\, d\phi = C_1 C_0[v_1(\overline{\phi})^\delta - v_1(\phi^*)^\delta]$, with $C_1 = \frac{\gamma}{[(\phi_{lb}^{FX})^{-\gamma} - (\phi_{ub}^{FX})^{-\gamma}][\frac{\beta}{(\sigma-1)} - B_1^A - \gamma]}$ and $\delta \equiv [\frac{\beta}{(\sigma-1)} - B_1^A - \gamma]/\frac{\beta}{(\sigma-1)} - B_1^A$.

We can summarize the results in this section as follows

$$\begin{aligned}
&v_0(\phi) \equiv \phi^{\frac{\beta\sigma}{(\sigma-1)}-B_1^A} g(\phi); \quad v_1(\phi) \equiv \phi^{\frac{\beta}{(\sigma-1)}-B_1^A}; \quad v(\phi) \equiv v_0(\phi)\left[1 + FX(\phi)\, n\tau^{1-\sigma}\right] \Rightarrow \\
&\Rightarrow V(H(\phi)) = B_0^V v(\phi); \quad \int_{\underline{s}}^{\overline{s}} V(s)\, ds = 1 \Rightarrow \quad B_0^V = \left[B_0^H \int_{\phi^*}^{\overline{\phi}} v(\phi)\, d\phi\right]^{-1} \\
&\int_{\phi^*}^{\overline{\phi}} v(\phi)\, d\phi = C_0\{[v_1(\overline{\phi}) - v_1(\phi^*)] + \cdots \\
&\qquad \cdots n\tau^{1-\sigma}[v_1(\overline{\phi})\, FX(\overline{\phi}) - v_1(\phi^*)\, FX(\phi^*) + C_1[v_1(\phi^*)^\delta - v_1(\overline{\phi})^\delta]]\}; \\
&C_0 \equiv \frac{\beta}{[\underline{\phi}^{-\beta} - \overline{\phi}^{-\beta}][\frac{\beta}{(\sigma-1)} - B_1^A]}; \quad C_1 = \frac{\gamma}{[(\phi_{lb}^{FX})^{-\gamma} - (\phi_{ub}^{FX})^{-\gamma}][\frac{\beta}{(\sigma-1)} - B_1^A - \gamma]}; \quad \delta \equiv \frac{\frac{\beta}{(\sigma-1)} - B_1^A - \gamma}{\frac{\beta}{(\sigma-1)} - B_1^A}.
\end{aligned} \tag{92}$$

C.3.3 Wage Schedule

The functional forms of $\{H(\phi), p(\phi)\}$ in (89) and (91) imply that, as a function of ϕ, $w(H(\phi))$ is of the following form

$$w(H(\phi)) = \frac{\sigma-1}{\sigma} A(H(\phi), \phi)\, p(\phi) = B_0^w w_0(\phi),$$

where $w_0(\phi) \equiv (\phi/\phi^*)^{B_1^A - \frac{\beta}{\sigma-1}}$ and B_0^w is constant.[67] In addition, B_0^w can be determined by the numeraire assumption ($\overline{w} = 1$), $B_0^W = 1/[B_0^H B_0^V \int_{\phi^*}^{\overline{\phi}} w_0(\phi) v(\phi)\, d\phi]$. Indeed, the functional forms of $w_0(\phi)$ above and of $v(\phi)$ in (92) imply that there is a closed-form expression for B_0^W, reducing the computational cost associated to numerical integration.

To compute $\int_{\phi^*}^{\overline{\phi}} w_0(\phi) v(\phi)\, d\phi$, it is convenient to break the integrand function into two terms,

$$w_0(\phi) v(\phi) = w_0(\phi) v_0(\phi) + w_0(\phi) v_0(\phi)\, FX(\phi)\, n\tau^{1-\sigma}.$$

Integrating the first term yields $\int_{\phi^*}^{\overline{\phi}} w_0(\phi) v_0(\phi)\, d\phi = C_0^w \left[\ln\left(\overline{\phi}\right) - \ln\left(\phi^*\right)\right]$, with $C_0^w \equiv \frac{(\phi^*)^{\frac{\beta}{\sigma-1} - B_1^A}\beta}{\left[\underline{\phi}^{-\beta} - \overline{\phi}^{-\beta}\right]}$. Integrating the second term yields

$$\int_{\phi_x^*}^{\overline{\phi}} w_0(\phi) v_0(\phi)\, FX(\phi)\, n\tau^{1-\sigma} d\phi = C_0^w n\tau^{1-\sigma} C_1^w \left\{ \frac{\left[\ln\left(\overline{\phi}\right) - \ln(\phi^*)\right]}{\left(\phi_{lb}^{FX}\right)^{\gamma}} + \frac{\left[\left(\overline{\phi}\right)^{-\gamma} - (\phi^*)^{-\gamma}\right]}{\gamma} \right\},$$

where $C_1^w \equiv \frac{1}{\left[\left(\phi_{lb}^{FX}\right)^{-\gamma} - \left(\phi_{ub}^{FX}\right)^{-\gamma}\right]}$.

We can summarize this discussion as follows

$$\begin{aligned}
& w_0(\phi) \equiv (\phi/\phi^*)^{B_1^A - \frac{\beta}{\sigma-1}}; \qquad v_0(\phi) \text{ and } v_1(\phi) \text{ as in (92)}; \\
& w(H(\phi)) = B_0^W w_0(\phi); \qquad \int_{\underline{s}}^{\overline{s}} w(s)\, V(s)\, ds = 1 \Rightarrow \quad B_0^W = \left[B_0^H B_0^V \int_{\phi^*}^{\overline{\phi}} w_0(\phi) v(\phi)\, d\phi\right]^{-1}; \\
& \int_{\phi^*}^{\overline{\phi}} w_0(\phi) v(\phi)\, d\phi = C_0^w \left\{ \left[\ln\left(\overline{\phi}\right) - \ln(\phi^*)\right] + n\tau^{1-\sigma} C_1^w \left[\frac{\left[\ln\left(\overline{\phi}\right) - \ln(\phi^*)\right]}{\left(\phi_{lb}^{FX}\right)^{\gamma}} + \frac{\left[\left(\overline{\phi}\right)^{-\gamma} - (\phi^*)^{-\gamma}\right]}{\gamma} \right] \right\}; \\
& C_0^w \equiv \frac{(\phi^*)^{\frac{\beta}{\sigma-1} - B_1^A}\beta}{\left[\underline{\phi}^{-\beta} - \overline{\phi}^{-\beta}\right]}; \qquad C_1^w \equiv \frac{1}{\left[\left(\phi_{lb}^{FX}\right)^{-\gamma} - \left(\phi_{ub}^{FX}\right)^{-\gamma}\right]};
\end{aligned} \tag{93}$$

C.3.4 Employment Schedule of Production Workers

Let $l^d(\phi)$ and $l^x(\phi)$ represent, respectively, the number of production workers employed by a firm with productivity ϕ to serve the domestic and foreign markets. The functional forms for $r^d(\phi)$ and $w(H(\phi))$ imply

$$\begin{aligned}
& l^d(\phi) = \frac{\sigma-1}{\sigma} \frac{r^d(\phi)}{w(H(\phi))} = (\sigma - 1) \frac{f(\phi/\phi^*)^{\frac{\beta\sigma}{\sigma-1} - B_1^A}}{B_0^w}; \\
& l^x(\phi) = l^d(\phi)\, n\tau^{1-\sigma} \text{ if } l^x(\phi) > 0; \\
& H_\phi(\phi), l_\phi^d(\phi) > 0 \Leftrightarrow \beta\sigma > B_1^A(\sigma - 1) > \beta
\end{aligned} \tag{94}$$

[67] Note that the condition $(\sigma - 1) B_1^A > \beta$ in (89) implies that $w(H(\phi))$ is an increasing function of ϕ.

where in the last expression I used the parameter restriction implied by $H_{\phi}(\phi) > 0$ in (89). Put another way, when the restriction on parameters in (94) holds, then the wage and the employment schedule are increasing functions of firm productivity ϕ.

C.3.5 CDF of Fixed Export Costs

The functional form assumptions on $\left\{r^{d}(\phi), FX(\phi)\right\}$ in (87) and (90) imply that the idiosyncratic part of fixed exports costs also have a truncated Pareto distribution. Recalling that F denotes the CDF of this distribution, our assumptions and the definitions of the model imply $FX(\phi) = F\left(\frac{fn\tau^{1-\sigma}}{f^{x}n}\left(\frac{\phi}{\phi^{*}}\right)^{\beta}\right)$. Letting $y \equiv \frac{fK_{1}}{f^{x}n}\left(\frac{\phi}{\phi^{*}}\right)^{\beta}$ for $\phi \in [\phi_{lb}^{FX}, \phi_{ub}^{FX}]$, we can change variables in the last equality to get

$$F(y) = \begin{cases} 0 & y < \underline{y} \\ \frac{[\underline{y}^{-\gamma/\beta} - y^{-\gamma/\beta}]}{[\underline{y}^{-\gamma/\beta} - \overline{y}^{-\gamma/\beta}]} & \text{if } \underline{y} \leq y \leq \overline{y} \\ 1 & \text{if } y > \overline{y} \end{cases} ;\ \text{with} \quad \begin{aligned} \underline{y} &= \frac{fn\tau^{1-\sigma}}{f^{x}n}\left(\phi_{lb}^{FX}/\phi^{*}\right)^{\beta} \\ \overline{y} &= \frac{fn\tau^{1-\sigma}}{f^{x}n}\left(\phi_{ub}^{FX}/\phi^{*}\right)^{\beta} \end{aligned} \tag{95}$$

C.3.6 Activity Cutoff and Entry Costs in the Free-Entry Model

The exogenous fixed cost of entry and the endogenous activity cutoff must jointly satisfy the free-entry condition (22). Integrating by parts the integral $\int y dF(y)$ in this expression, and later changing the order of integration in the resulting expression, the free-entry condition (22) can be expressed as

$$\int_{\phi^{*}}^{\overline{\phi}} f(\phi/\phi^{*})^{\beta}[1 - G(\phi)]\frac{\beta}{\phi}\left[1 + FX(\phi)n\tau^{1-\sigma}\right]d\phi = f_{e} \tag{96}$$

As I discuss in section C.4, I choose $\left\{\phi^{*}, \overline{\phi}, f\right\}$ as normalizations, implying that the last expression can be used to recover f_{e}.

C.3.7 Activity Cutoff and Mass of Firms in the No-Free-Entry Model

The exogenous mass of firms and the endogenous activity cutoff must jointly satisfy condition (19) in the no-free-entry model. Noting that the wage of nonproduction workers can be expressed as the difference between operational (or variable) profits and total profits, the same calculations leading to (96) imply that condition (19) can be expressed as

$$\int_{\phi^{*}}^{\overline{\phi}} f(\phi/\phi^{*})^{\beta}[1 + FX(\phi)n\tau^{1-\sigma}]g(\phi)\left[\sigma - \frac{[1-G(\phi)]}{g(\phi)}\frac{\beta}{\phi}\right]d\phi = \frac{L}{M}. \tag{97}$$

As I discuss in section C.4, I choose $\left\{\phi^*, \overline{\phi}, f\right\}$ as normalizations and obtain L from the data, implying that the last expression can be used to recover $\overline{M}$.

C.4 Computing Relevant Empirical Moments in the Model

The assumptions and results from the sections C.1-C.3 allow me to derive the distribution of value added per worker in the model as well as the model's implications for other relevant moments in the data. In particular, the expressions I derive below can be used to compute the model's predictions for the empirical moments targeted in the estimation strategy described later.

C.4.1 Distribution of value added per worker

An **exporter** exhibits value added per worker below z if $XVA\left(\phi, y\right) \equiv \frac{r^d(\phi)\left[1+n\tau^{1-\sigma}\right]}{l^d(\phi)+l^x(\phi)+f+nf^x y} \leq z$, or alternatively, if

$vax\left(\phi, z\right) \equiv \frac{r^d(\phi)\left[1+n\tau^{1-\sigma}\right]}{znf^x} - \frac{\left[l^d(\phi)+l^x(\phi)+f\right]}{nf^x} \leq y.$

That is, $vax\left(\phi, z\right)$ represents the cutoff value of the idiosyncratic fixed export cost y above which all exporters with productivity ϕ exhibit a value added per worker below z. In addition, an active firm with productivity ϕ is an exporter if and only if $\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma} \geq f^x y$, or

$ix\left(\phi\right) \equiv \frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x} \geq y.$

Accordingly, a firm with productivity ϕ exports and has value added below z if and only if its idiosyncratic export cost y satisfies $vax\left(\phi, z\right) \leq y \leq ix\left(\phi\right)$. Letting $fvax\left(\phi, z\right) \equiv \max\left\{F\left[ix\left(\phi\right)\right] - F\left[vax\left(\phi, z\right)\right], 0\right\}$, these observations imply that the share of all firms that export and exhibit value added per worker below z is given by

$GXVA\left(z\right) = \int_{\phi^*}^{\overline{\phi}} fvax\left(\phi, z\right) \frac{g(\phi)}{\left[1-G(\phi^*)\right]} d\phi.$

What are the lower and upper bounds of the distribution implied by $GXVA\left(z\right)$? The value added per worker of an exporter with a vector of characteristics $\left(\phi, y\right)$, $XVA\left(\phi, y\right)$, is decreasing in its idiosyncratic fixed export cost y. Accordingly, the minimum and maximum value of $XVA\left(\phi, y\right)$ among exporters with productivity ϕ are achieved at $y_h^x\left(\phi\right) = \frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}$ and $y_l^x = \underline{y}$, respectively. In addition, some algebraic manipulation yields $XVA\left(\phi, y_h^x\left(\phi\right)\right) = \left[1 + n\tau^{1-\sigma}\right] / \left[\frac{(\sigma-1)}{\sigma w(H(\phi))} + \frac{f}{r^d(\phi)} + \frac{n\tau^{1-\sigma}}{\sigma}\right]$, which implies that $XVA\left(\phi, y_h^x\left(\phi\right)\right)$ is increasing in ϕ as $w\left(H\left(\phi\right)\right)$ and $r^d\left(\phi\right)$ are.[68] A similar calculation shows that $XVA\left(\phi, y_l^x\right)$ is also increasing in ϕ. These observations imply $z_l^x = XVA\left(\phi_x^*, y_h^x\left(\phi_x^*\right)\right)$ and $z_h^x = XVA\left(\overline{\phi}, y_l^x\right)$, where ϕ_x^* is the productivity level above which the fraction of exporters is positive.

Turning to **nonexporters**, a firm with productivity ϕ that serves only its domestic market has value added weakly below z if and only if $DVA\left(\phi\right) \equiv \frac{r^d(\phi)}{l^d(\phi)+f} \leq z$. Letting ϕ^z be the productivity value such that $DVA\left(\phi^z\right) = z$, the fraction of total firms that are nonexporters and have value added below z is given by,

$GDVA\left(z\right) = \int_{\phi^*}^{\phi^z} \left(1 - F\left[ix\left(\phi\right)\right]\right) \frac{g(\phi)}{\left[1-G(\phi^*)\right]} d\phi,$

[68] This derivation uses the expressions for $l^d\left(\phi\right)$ and $l^x\left(\phi\right)$ in (94).

where $F\left[ix\left(\phi\right)\right]=FX\left(\phi\right)$ is just the fraction of exporters among firms with productivity ϕ. Moreover, our functional form assumption for $FX\left(\phi\right)$ implies that we can compute that last integral in close form, $GDVA\left(z\right)=C_0\int_{\phi^*}^{\phi^z}[\phi^{-\gamma-\beta-1}-\left(\phi_{ub}^{FX}\right)^{-\gamma}\phi^{-\beta-1}]d\phi$, with $C_0\equiv\frac{\beta}{[(\phi_x^*)^{-\gamma}-\left(\phi_{ub}^{FX}\right)^{-\gamma}][(\phi^*)^{-\beta}-(\overline{\phi})^{-\beta}]}$.

In addition, note that the value added per worker of a nonexporter with productivity ϕ can be expressed as $DVA\left(\phi\right)=1/[\frac{(\sigma-1)}{\sigma w(H(\phi))}+\frac{f}{r^d(\phi)}]$. As $\left\{w\left(H\left(\phi\right)\right),r^d\left(\phi\right)\right\}$ are increasing in ϕ, so is $DVA\left(\phi\right)$. As such, the minimum and maximum value added per worker among nonexporters is given by $z_l^d=DVA\left(\phi^*\right)$ and $z_h^d=DVA\left(\overline{\phi}\right)$.

Finally, the distribution of value added per worker among all active firms is given by $GVA(z)=GXVA\left(z\right)+GDVA\left(z\right)$, with the lower bound of the distribution given by $z_l=\min\left\{z_l^d,z_l^x\right\}$ and the upper bound by $z_h=\min\left\{z_h^d,z_l^x\right\}$.

The discussion in this section can be summarized as follows

$$
\begin{array}{ll}
XVA\left(\phi,y\right)\equiv\frac{r^d(\phi)\left[1+n\tau^{1-\sigma}\right]}{l^d(\phi)+l^x(\phi)+f+nf^xy}; & vax\left(\phi,z\right)\equiv\frac{r^d(\phi)\left[1+n\tau^{1-\sigma}\right]}{znf^x}-\frac{\left[l^d(\phi)+l^x(\phi)+f\right]}{nf^x}; \\
ix\left(\phi\right)\equiv\frac{r^d(\phi)\tau^{1-\sigma}}{\sigma f^x}; & fvax\left(\phi,z\right)\equiv\max\left\{F\left[ix\left(\phi\right)\right]-F\left[vax\left(\phi,z\right)\right],0\right\}; \\
GXVA\left(z\right)=\int_{\phi^*}^{\overline{\phi}}fvax\left(\phi,z\right)\frac{g(\phi)}{[1-G(\phi^*)]}d\phi; & z_l^x=XVA\left(\phi_x^*,y_h^x\left(\phi_x^*\right)\right);\ z_h^x=XVA\left(\overline{\phi},y_l^x\right); \\
DVA\left(\phi\right)\equiv\frac{r^d(\phi)}{l^d(\phi)+f}; & \phi^z\text{ solves }DVA\left(\phi^z\right)=z; \\
GDVA\left(z\right)=\int_{\phi^*}^{\phi^z}\left(1-F\left[ix\left(\phi\right)\right]\right)\frac{g(\phi)}{[1-G(\phi^*)]}d\phi; & GDVA\left(z\right)=C_0\int_{\phi^*}^{\phi^z}[\phi^{-\gamma-\beta-1}-\left(\phi_{ub}^{FX}\right)^{-\gamma}\phi^{-\beta-1}]d\phi \\
C_0\equiv\frac{\beta}{[(\phi_x^*)^{-\gamma}-\left(\phi_{ub}^{FX}\right)^{-\gamma}][(\phi^*)^{-\beta}-(\overline{\phi})^{-\beta}]} & z_l^d=DVA\left(\phi^*\right);\ z_h^d=DVA\left(\overline{\phi}\right) \\
GVA(z)=GXVA\left(z\right)+GDVA\left(z\right) & z_l=\min\left\{z_l^d,z_l^x\right\};\ z_h=\min\left\{z_h^d,z_l^x\right\}
\end{array}
\tag{98}
$$

C.4.2 Average value added per worker in each decile

The average value added per worker in decile i is given by

$$aVA(i)=\int_{z_{i-1}}^{z_i}zdGVA(z)/0.1=\left\{[zGVA(z)|_{z_{i-1}}^{z_i}-\int_{z_{i-1}}^{z_i}GVA(z)dz\right\}/0.1 \tag{99}$$

C.4.3 Fraction of firms that are exporters in each decile

Let z_i be the upper bound of decile i of the distribution of firms in terms of value added per worker. Noting that $GVA(z_i)-GVA(z_{i-1})=0.1$, then the fraction of firms that export in each decile, $IX\left(i\right)$, is given by

$$IX\left(i\right)=\frac{GXVA\left(z_{i}\right)-GXVA\left(z_{i-1}\right)}{0.1}\tag{100}$$

C.4.4 Total employment in each decile

I start by computing the total mass of workers employed by **exporters** with value added per worker below some level z—i.e., those firms whose idiosyncratic fixed export costs y satisfies $vax\left(\phi,z\right)\leq y\leq ix\left(\phi\right)$. Letting $\mathcal{X}\left(z\right)\equiv\left\{\phi\in\left[\phi^{*},\overline{\phi}\right]:vax\left(\phi,z\right)\leq ix\left(\phi\right)\right\}$, the total mass of production workers employed at exporters with value added weakly lower than z is defined as
$LVAX\left(z\right)=\int_{\mathcal{X}(z)}\int_{vax(\phi,z)}^{ix(\phi)}\left[l^{d}\left(\phi\right)+l^{x}\left(\phi\right)+f+nf^{x}y\right]dFg\left(\phi\right)d\phi\overline{M}$,
which after some manipulation yields

$$LVAX\left(z\right)=\int_{\phi^{*}}^{\overline{\phi}}fvax\left(\phi,z\right)\left(\begin{array}{c}\frac{\sigma-1}{\sigma}\frac{r^{d}(\phi)\left[1+n\tau^{1-\sigma}\right]}{w(H(\phi))}+\\ f+nf^{x}E\left[y|y\in\left[vax\left(\phi,z\right),ix\left(\phi\right)\right]\right]\end{array}\right)g\left(\phi\right)d\phi\overline{M}.$$

Although the last expression is helpful conceptually, for computational purposes it is convenient to differentiate between production and nonproduction workers. Starting with some notation, of the mass of all workers employed at exporters with value added per worker weakly below z, $LVAX\left(z\right)$, $LPVAX(z)$ denotes production workers, $LNPf^{x}VAX(z)$ denotes nonproduction workers related to fixed export costs, and $LNPfVAX$ denotes nonproduction workers related to fixed costs of production. With these definitions, and letting $1_{\mathcal{X}(z)}$ be the indicator function of the set $\mathcal{X}\left(z\right)$, we get

$$LPVAX(z)\equiv\int_{\phi^{*}}^{\overline{\phi}}fvax\left(\phi,z\right)\left[l^{d}\left(\phi\right)+l^{x}\left(\phi\right)\right]g\left(\phi\right)d\phi\overline{M},$$

$$LNPf^{x}VAX(z)\equiv nf^{x}\int_{\phi^{*}}^{\overline{\phi}}1_{\mathcal{X}(z)}\int_{vax(\phi,z)}^{ix(\phi)}ydFg\left(\phi\right)d\phi\overline{M},$$

$$LNPfVAX(z)\equiv f\int_{\phi^{*}}^{\overline{\phi}}fvax\left(\phi,z\right)g\left(\phi\right)d\phi\overline{M}=f\times M\times GXVA(z).$$

From a computational point of view, a more convenient expression for $LNPf^{x}VAX(z)$ can be obtained using the functional form for F in (95),

$$LNPf^{x}VAX(z)=C_{1}\int_{\phi^{*}}^{\overline{\phi}}\int_{vax(\phi,z)}^{ix(\phi)}y^{-\gamma/\beta}dyg\left(\phi\right)d\phi,\text{ with }C_{1}=\frac{nf^{x}\gamma\overline{M}}{\beta\left[\underline{y}^{-\gamma/\beta}-\overline{y}^{-\gamma/\beta}\right]},$$

with the convention that $\int_{vax(\phi,z)}^{ix(\phi)}y^{-\gamma/\beta}dy=0$ if $ix\left(\phi\right)<vax\left(\phi,z\right)$. Note that, as the integral of a power function, $\int_{vax(\phi,z)}^{ix(\phi)}y^{-\gamma/\beta}dy$ has a closed-form expression as a function of the limits of integration, so it does not require numerical integration. I use numerical integration only for the outer integral $\int_{\phi_{x}^{*}}^{\overline{\phi}}.d\phi$.

Similarly, the total mass of workers employed by **nonexporters** with value added per worker weakly below z, $LVAD\left(z\right)$, is the sum of production and nonproduction workers, $LVAD\left(z\right)=LPVAD\left(z\right)+LNPfVAD\left(z\right)$, with

$$LPVAD\left(z\right)=\int_{\phi^{*}}^{\phi^{z}}\left(1-F\left[ix\left(\phi\right)\right]\right)l^{d}\left(\phi\right)g\left(\phi\right)d\phi\overline{M}.$$

$$LNPfVAD\left(z\right)=\int_{\phi^{*}}^{\phi^{z}}\left(1-F\left[ix\left(\phi\right)\right]\right)fg\left(\phi\right)d\phi\overline{M}=f\times M\times GDVA(z).$$

The functional form for F implies

$LPVAD\left(z\right)=C_{0}\int_{\phi^{*}}^{\phi^{z}}[\phi^{\frac{\beta}{\sigma-1}-B_{1}^{A}-1-\gamma}-\left(\phi_{ub}^{FX}\right)^{-\gamma}\phi^{\frac{\beta}{\sigma-1}-B_{1}^{A}-1}]d\phi,$
with $C_{0}=\frac{(\sigma-1)f\beta\overline{M}}{(\phi^{*})^{\frac{\beta\sigma}{\sigma-1}-B_{1}^{A}}B_{0}^{w}[\left(\phi_{lb}^{FX}\right)^{-\gamma}-\left(\phi_{ub}^{FX}\right)^{-\gamma}][\underline{\phi}^{-\beta}-\overline{\phi}^{-\beta}]}.$

The integrand in the last expression is just a sum of power functions, so the value of the integral has a closed-form expression as a function of the limits of integration. As discussed above, this eliminates the need of numerical integration, reducing computation time significantly.

Finally, letting z_i denote the upper bound of decile i of the distribution of of value added per worker, then the total mass of workers employed by firms in decile i, $L^{tot}\left(i\right)$, is given by $L^{tot}\left(i\right)=LVA\left(z_{i}\right)-LVA\left(z_{i-1}\right)$.

The discussion in this section is summarized below,

$$
\begin{array}{ll}
\mathcal{X}\left(z\right)=\left\{\phi\in\left[\phi^{*},\overline{\phi}\right]:vax\left(\phi,z\right)\leq ix\left(\phi\right)\right\}; & M=\left[1-G\left(\phi^{*}\right)\right] \\
LPVAX(z)=\int_{\phi^{*}}^{\overline{\phi}}fvax\left(\phi,z\right)\left[l^{d}\left(\phi\right)+l^{x}\left(\phi\right)\right]g\left(\phi\right)d\phi\overline{M}; & \\
LNPf^{x}VAX(z)=C_{1}\int_{\phi^{*}}^{\overline{\phi}}\int_{vax(\phi,z)}^{ix(\phi)}y^{-\gamma/\beta}dyg\left(\phi\right)d\phi; & C_{1}=\frac{nf^{x}\gamma\overline{M}}{\beta[\underline{y}^{-\gamma/\beta}-\overline{y}^{-\gamma/\beta}]}; \\
LNPfVAX(z)=f\times M\times GXVA(z); & \\
LVAX\left(z\right)=LPVAX(z)+LNPf^{x}VAX(z)+LNPfVAX(z); & \\
LPVAD\left(z\right)=C_{0}\int_{\phi^{*}}^{\phi^{z}}[\phi^{\frac{\beta}{\sigma-1}-B_{1}^{A}-1-\gamma}-\left(\phi_{ub}^{FX}\right)^{-\gamma}\phi^{\frac{\beta}{\sigma-1}-B_{1}^{A}-1}]d\phi; & DVA\left(\phi^{z}\right)=z; \\
C_{0}=\frac{(\sigma-1)f\beta\overline{M}}{(\phi^{*})^{\frac{\beta\sigma}{\sigma-1}-B_{1}^{A}}B_{0}^{w}[\left(\phi_{lb}^{FX}\right)^{-\gamma}-\left(\phi_{ub}^{FX}\right)^{-\gamma}][\underline{\phi}^{-\beta}-\overline{\phi}^{-\beta}]}; & \\
LNPfVAD\left(z\right)=f\times M\times GDVA(z); & \\
LVA\left(z\right)\equiv LVAD\left(z\right)+LVAX\left(z\right); & \\
L^{tot}\left(i\right)=LVA\left(z_{i}\right)-LVA\left(z_{i-1}\right),\text{ for decile }i. &
\end{array}
\tag{101}
$$

C.4.5 Total wages paid in each decile

Total wages paid by **exporters** with value added per worker no greater than z ($WVAX(z)$) equals the sum of the wages they pay to production workers ($WPVAX(z)$), to nonproduction workers associated with fixed export costs ($WNPfxVAX(z)$), and to nonproduction workers associated with fixed production costs ($WNPfVAX(z)$), $WVAX(z)=WPVAX(z)+WNPfxVAX(z)+WNPfVAX(z)$. Similarly, total wages paid by **nonexporters** with value added per worker no greater than z ($WVAD(z)$) equals the sum of the wages they pay to production workers ($WPVAD(z)$) and to nonproduction workers associated fixed production costs ($WNPfVAD(z)$), $WVAD(z) = WPVAD(z) + WNPfVAD(z)$. The analysis of the previous section, together with the numeraire assumption, $\overline{w} = 1$, yields the following expressions

$$\boxed{\begin{array}{l} WPVAX(z) = \int_{\phi^*}^{\overline{\phi}} fvax\left(\phi, z\right) w\left(H\left(\phi\right)\right)\left[l^d\left(\phi\right) + l^x\left(\phi\right)\right] g\left(\phi\right) d\phi \overline{M}; \\ \\ WNPfxVAX(z) = LNPf^xVAX(z); \qquad WNPfVAX(z) = LNPfVAX(z); \\ \\ WVAX(z) = WPVAX(z) + WNPfxVAX(z) + WNPfVAX(z) \\ \\ WPVAD\left(z\right) = \int_{\phi^*}^{\phi^z} \left(1 - F\left[ix\left(\phi\right)\right]\right) w\left(H\left(\phi\right)\right) l^d\left(\phi\right) g\left(\phi\right) d\phi \overline{M};\ \phi^z \text{ solves } DVA\left(\phi^z\right) = z; \\ \\ WNPfVAD(z) = LNPfVAD\left(z\right); \qquad WVAD(z) = WPVAD(z) + WNPfVAD(z). \end{array}} \tag{102}$$

C.4.6 Some comments

As we can see from (98)-(102), these expressions ultimately depend on some of the parameters of $\{G\left(\phi\right), r^d\left(\phi\right), A\left(s, \phi\right), FX\left(\phi\right)\}$ and on the upper and lower bounds of the productivity distribution of active firms $\left\{\phi^*, \overline{\phi}\right\}$. Although some of these expressions depend on fixed costs $\{f, f_x\}$, such as the one for the labor schedule in (94), I show in the next section that these parameters do not affect the model implications for the moments targeted in the estimation.

C.5 Main Estimation: Fitting Moments in Portuguese Data

The main (and more involved) estimation exercise in the paper aims at making the model fit relevant moments in the firm data from Portugal described in the main text. Below, I discuss the moments of the data selected as target of the estimation, the parameters that affect the value of these moments in the model, and the estimation method.

C.5.1 Targeted Moments, Relevant Parameters and Estimation Method

The moments in the data that are targeted in the estimation are selected based on data availability and on their informational content about key elements of the model. Specifically, with firm data broken down by decile of value added per worker, the estimation targets (i) the distribution of total employment across deciles, (ii) distribution of the total wage bill across deciles, (iii) the fraction of firms that export in each decile, and (iv) the average value added per worker in each decile. Moment (iii) is a crucial taget of the estimation because of its informational content about the extensive-margin channel in the model, the channel driving much of theoretical ambiguity regarding the distributional effects of trade.

The values of the targeted moments implied by the model follow immediately from expressions (98)-(102), which are the basis of the estimation algorithm. These expressions and those in (86)-(97) indicate which model parameters are relevant for these moments. In particular, these moments depend on the bounds of the support of the productivity distribution of active firms $\{\phi^*, \overline{\phi}\}$, parameters of $\{G(\phi), r^d(\phi)\}$, β $(= \theta)$, the parameter B_1^A of the productivity function $A(s, \phi)$, the parameters of the function $FX(\phi)$, $\{\phi_{lb}^{FX},\ \phi_{ub}^{FX},\ \gamma\}$, the elasticity of substitution σ and the level of trade costs and number of symmetric countries, $\{\tau, n\}$. The other parameters of the productivity function $\{\alpha_s,\ \alpha_s,\ \rho,\ B_0^A\}$ do not affect the model-implied values for the targeted moments. Moreover, although the fixed costs parameters $\{f, f_x\}$ appear in some of these expressions, I show in the section C.6 that they do not affect the moments targeted in the estimation nor the wage distribution in the calibrated equilibrium. I also show that the parameters $\{\tau, n\}$ affect relevant variables variables only through the value of $n\tau^{1-\sigma}$. All parameters that are not directly relevant for the targeted moments are assigned normalized values in the calculations. As the selection of $\{\phi^*, \overline{\phi}\}$ is equivalent to a choice of measurement units for firm productivity, I also normalize the values of these parameters in the estimation.

Given these normalizations and the values for $\{\sigma, n\tau^{1-\sigma}\}$ pinned down in (86), the estimation needs to find values for the parameters $\{\phi_{lb}^{FX},\ \phi_{ub}^{FX},\ \gamma,\ \beta,\ B_1^A\}$ that best fit the target moments. A comment is in order about the selection of the parameters of the function $FX(\phi)$, $\{\phi_{lb}^{FX},\ \phi_{ub}^{FX},\ \gamma\}$. It is readily seen that the only thing that matters for the estimation is the values that the function $FX(\phi)$ takes on the interval $[\phi^*, \overline{\phi}]$. In addition, given the functional form assumed for $FX(\phi)$, these values are completely determined once we know γ and the value that the function FX takes at some $\widetilde{\phi} \in [\phi^*, \overline{\phi}]$, $FX(\widetilde{\phi})$. Accordingly, in my estimation I choose $\{\widetilde{\phi},\ FX(\widetilde{\phi})\}$ instead of $\{\phi_{lb}^{FX},\ \phi_{ub}^{FX}\}$, recovering the latter from the estimated values of the former.

Finally, I estimate $\{\widetilde{\phi},\ FX(\widetilde{\phi}),\ \gamma,\ \beta,\ B_1^A\}$ by the method of simulated moments, using as target the moments (i)-(iv) described above.

C.6 Model Parameters, Targeted Moments, and the Calibrated Equilibrium

In this section, I show that the fixed costs parameters $\{f, f_x\}$ do not affect the moments targeted in the estimation described in section C.5 nor the wage distribution in the calibrated equilibrium. I also show that the parameters $\{\tau, n\}$ affect relevant variables variables only through the value of $n\tau^{1-\sigma}$. Finally, I show that the calibrated CDF of fixed export costs satisfies the sufficient condition in proposition 3.iii.

C.6.1 Parameters and Targeted Moments in the Model

Given the calibration strategy discussed in section C.5, the values selected for the fixed cost parameters $\{f, f_x\}$ do not affect the value of the targeted moments in the model. This result largely reflects the approach of imposing a functional form directly on the endogenous fraction of firms that exporter at each productivity level, $FX(\phi)$, and later recovering the CDF of exports costs as residuals as indicated in (95).

As a first step, note that the CDF of value added per worker in the calibrated model, $GVA(z)$, is not affected by these parameters. As shown in (98), $GVA(z) = GXVA(z) + GDVA(z)$, where $GXVA(z)$ and $GDVA(z)$ are, respectively, the share of all firms that are exporters and nonexporters and that exhibit value added per worker below z. The expression for $GDVA(z)$ in (98) and those for $\{r^d(\phi), l^d(\phi), F\}$ in

(94)-(95) imply

$$GDVA(z) = \int_{\phi^*}^{\phi^z} [1 - FX(\phi)] \frac{g(\phi)}{[1 - G(\phi^*)]} d\phi, \text{ with } \frac{\sigma(\phi^z/\phi^*)^\beta B_0^w}{(\sigma - 1)(\phi^z/\phi^*)^{\frac{\beta\sigma}{\sigma-1} - B_1^A} + B_0^w} = z. \tag{103}$$

In addition, the results in (92) and (93) imply $B_0^w = \int_{\phi^*}^{\overline{\phi}} v(\phi)\, d\phi / \int_{\phi^*}^{\overline{\phi}} w_0(\phi) v(\phi)\, d\phi$, an expression that does not depend on $\{f, f_x\}$ either.

Turning to $GXVA(z)$, according to the expression in (98), $GXVA(z)$ could depend on $\{f, f_x\}$ only through $F[vax(\phi, z)]$ in the definition of the function $fvax(\phi, z)$. However, the definitions of $\{F, vax(\phi, z)\}$ in (95) and (98) imply

$$F[vax(\phi, z)] = \widetilde{F}[\widetilde{vax}(\phi, z)], \text{ where}$$

$$\widetilde{vax}(\phi, z) \equiv \frac{\sigma(\phi/\phi^*)^\beta [1 + n\tau^{1-\sigma}]}{z} - \left[\frac{(\sigma-1)(\phi/\phi^*)^{\frac{\beta\sigma}{\sigma-1} - B_1^A} [1 + n\tau^{1-\sigma}]}{B_0^w} + 1 \right], \tag{104}$$

$$\widetilde{F}(u) \equiv \begin{cases} 0 & u < \underline{u} \\ \frac{[\underline{u}^{-\gamma/\beta} - y^{-\gamma/\beta}]}{[\underline{u}^{-\gamma/\beta} - \overline{u}^{-\gamma/\beta}]} & \text{if } \underline{u} \le u \le \overline{u} \\ 1 & \text{if } u > \overline{u} \end{cases} ; \text{ with } \begin{array}{l} \underline{u} = n\tau^{1-\sigma} \left(\phi_{lb}^{FX}/\phi^*\right)^\beta, \\ \\ \overline{u} = n\tau^{1-\sigma} \left(\phi_{ub}^{FX}/\phi^*\right)^\beta, \end{array}$$

so $F[vax(\phi, z)]$ does not depend on the fixed costs parameters $\{f, f_x\}$.

The previous derivations shows that $GVA(z)$ does not depend on $\{f, f_x\}$. In turn, this result and equations (99) and (100) immediately imply that the average value added per worker and the fractions of firms that export in each decile of value added per worker do not depend on these parameters either. Similar derivations using the expressions in (101) and (102) also show that the distribution of total employment and the total wage bill across deciles of value added per worker do not depend on $\{f, f_x\}$.

I now turn to the dependence of the moments targeted in the estimation on the parameters $\{\tau, n\}$. Expressions (92)-(93) and (103)-(104) imply that the CDF of value added per worker in the calibrated model, $GVA(z)$, depends on $\{\tau, n\}$ only through $n\tau^{1-\sigma}$. As discussed earlier, this result, together with similar calculations using (101) and (102), implies that all targeted moments depends on $\{\tau, n\}$ only through $n\tau^{1-\sigma}$.

C.6.2 Wage Distribution in the Calibrated Equilibrium

Given the calibration strategy discussed in section C.5, the values of the fixed cost parameters $\{f, f_x\}$ do not affect the wage distribution (Lorenz curve) in the calibrated equilibrium. In addition, the values of the parameters $\{n, \tau\}$ affect the distribution only through the value of $n\tau^{1-\sigma}$.

As discussed in section B.1.2 of the appendix, the Lorenz curve of wage income maps the fraction ξ of poorest workers in the economy to the fraction $\mathcal{L}(\xi)$ of the total wage income in the economy accruing to these workers. Formally, for each $\xi \in [0, 1]$, let $s(\xi)$ be the skill level that solves $\xi = \int_{\underline{s}}^{s(\xi)} V(s)\, ds$.

Then, $\mathcal{L}(\xi) = \int_{\underline{s}}^{s(\xi)} w(s)V(s)\,ds / \int_{\underline{s}}^{\overline{s}} w(s)V(s)\,ds$. Per the numeraire assumption, we can write $\mathcal{L}(\xi) = \int_{\underline{s}}^{s(\xi)} w(s)V(s)\,ds$. To get the desired result, it is convenient to change the variable of integration in the previous expressions. Specifically, if we define $\phi(\xi) \equiv N(s(\xi))$, then $\phi(\xi)$ solves

$$\xi = \int_{\phi^*}^{\phi(\xi)} V(H(\phi))\, B_0^H d\phi = \frac{\int_{\phi^*}^{\phi(\xi)} v(\phi)\,d\phi}{\int_{\phi^*}^{\overline{\phi}} v(\phi)\,d\phi},$$

where in the last derivation I used the expressions in (92). Similarly,

$$\mathcal{L}(\xi) = \frac{\int_{\phi^*}^{\phi(\xi)} w_0(\phi)\, v(\phi)\,d\phi}{\int_{\phi^*}^{\overline{\phi}} w_0(\phi)\, v(\phi)\,d\phi}.$$

The last two expressions, together with (92) and (93), imply that $\phi(\xi)$ and $\mathcal{L}(\xi)$ are not affected by the values assigned to the parameters $\{f, f_x\}$. Finally, these expressions also show that $\{n, \tau\}$ appear in the definitions of $v(\phi)$ and $w_0(\phi)$ only through $n\tau^{1-\sigma}$.

C.6.3 CDF of Fixed Export Costs

The calibrated CDF of fixed export costs, $F(y)$, satisfies the sufficient condition in proposition 3.iii. To see this, note that the functional form for $F(y)$ in (95) implies that the functions $\eta_0^F(t,\lambda) \equiv \frac{F_y(t\lambda)\lambda}{[1+F(t\lambda)k]}$ and $\eta_1^F(t,\lambda) \equiv \frac{F_y(t\lambda)\lambda^2}{[1+F(t\lambda)k]}$ are given by

$$\eta_0^F(t,\lambda) = \frac{\gamma t^{-\nu-1}\lambda^{-\nu}}{[\underline{y}^{-\nu} - \overline{y}^{-\nu}] + [\underline{y}^{-\nu} - (t\lambda)^{-\nu}]k},$$
$$\eta_1^F(t,\lambda) = \frac{\gamma t^{-\nu-1}\lambda^{1-\nu}}{[\underline{y}^{-\nu} - \overline{y}^{-\nu}] + [\underline{y}^{-\nu} - (t\lambda)^{-\nu}]k},$$

where $\nu \equiv \gamma/\beta \approx 0.24$. It is readily seen the function $\eta_0^F(t,\lambda)$ is strictly decreasing in λ for all relevant values of (t,k).

$$\eta_1^F(t,\lambda) = \frac{\gamma t^{-\nu-1}\lambda^{1-\nu}}{[\underline{y}^{-\nu} - \overline{y}^{-\nu}] + [\underline{y}^{-\nu} - (t\lambda)^{-\nu}]k}.$$

To see if η_1^F is increasing or not, I will compute the growth rate of the numerator and denominator of η_1^F as λ increases, which I denote by $\widehat{num}_1^F$ and $\widehat{den}_1^F$, respectively. Then
$\widehat{num}_1^F = (1-\nu)\widehat{\lambda}$
and
$$\widehat{den}_1^F(\lambda t) = \frac{(t\lambda)^{-\nu}\nu\widehat{\lambda}}{[\underline{y}^{-\nu} - \overline{y}^{-\nu}] + [\underline{y}^{-\nu} - (t\lambda)^{-\nu}]k} = \frac{\frac{(t\lambda)^{-\nu}}{[\underline{y}^{-\nu} - \overline{y}^{-\nu}]}\nu\widehat{\lambda}}{1 + F(\lambda t)\,k}.$$

Note that the expression for $\widehat{den}_1^F$ is decreasing in λt, so for any relevant value of k. As such, if $\widehat{num}_1^F > \widehat{den}_1^F(\lambda t)$ for the minimum possible value of λt (where $\widehat{den}_1^F$ attains its maximum value), then

$\eta_1^F(t,\lambda)$ is increasing in λ for $\lambda \geq 1$ and all relevant values of t and k. At the minimum value of λt, $\underline{y}$, the calibrated model implies

$$\widehat{den}_1^F(\underline{y}) \approx 0.27 \times \widehat{\lambda} < 0.76 \times \widehat{\lambda} \approx \widehat{num}_1^F,$$

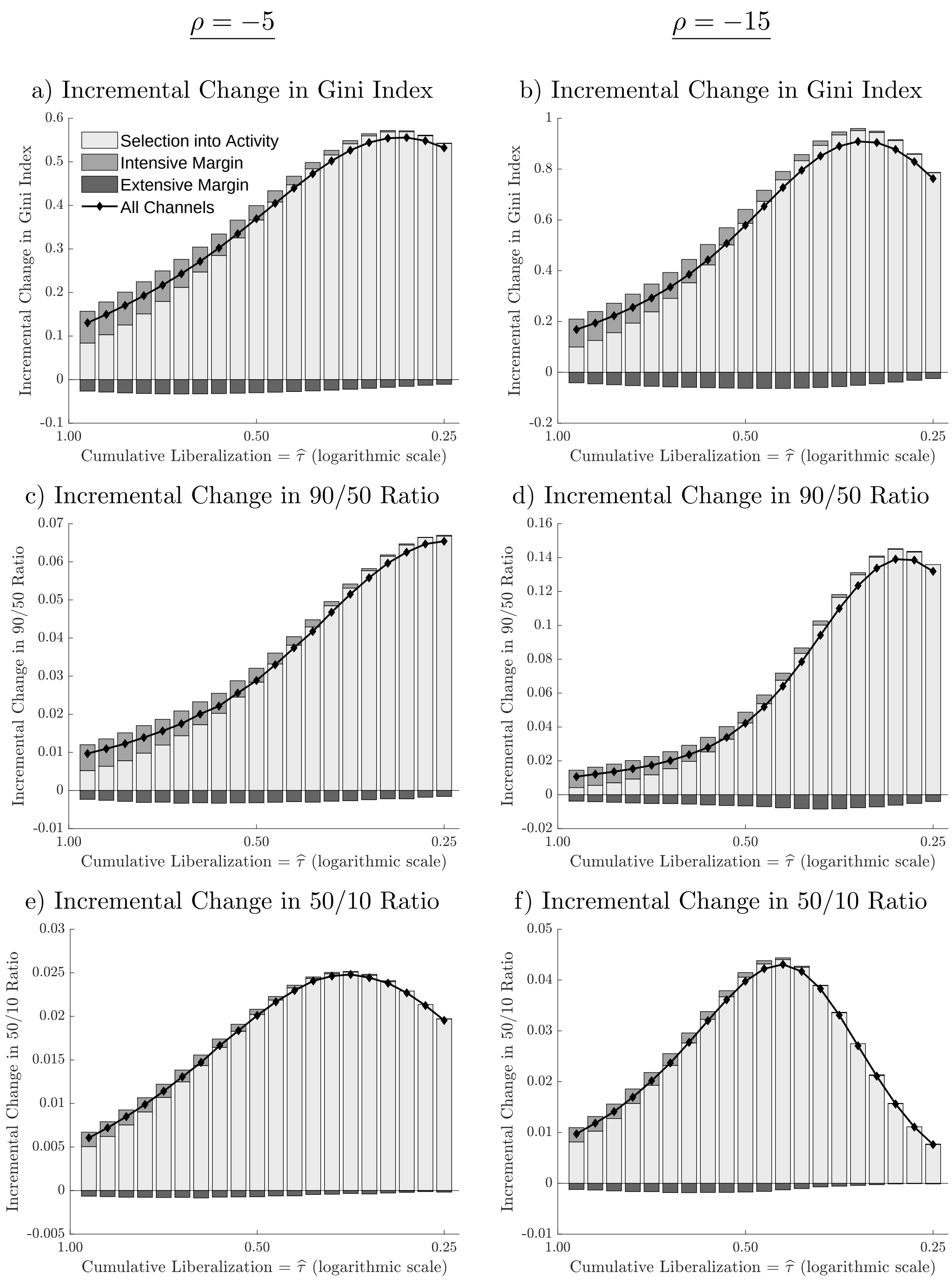
where the last expression uses $\frac{\underline{y}^{-\nu}}{[\underline{y}^{-\nu} - \overline{y}^{-\nu}]} \approx 1.12$, $\nu \approx 0.24$ and $(1-\nu) \approx 0.76$. These results imply that η_1^F is increasing in λ for all relevant values of t and k.

D Sensitivity Analysis: Skill-Productivity Substitutability

Despite not affecting wage inequality in the calibrated equilibrium, the elasticity of substitution between worker skill and firm productivity in the productivity function $A(s,\phi)$, ρ, does affect the distributional effects of changes in trade costs. As discussed in section 7, for lower values of ρ (s and ϕ are harder to substitute), a given change in trade costs leads to larger changes in wage inequality, as larger changes in relative wages are required for firms to change their optimal choice of worker type. That said, as the CDF of fixed export costs satisfies the sufficient condition in proposition 3.iii, the broad qualitative effect of a decline in trade costs is always a pervasive rise in wage inequality, regardless of the value of ρ.

Figure 12 illustrates these results by repeating the analysis of figures 5 and 7 for two alternative values of ρ, $\rho = -5, -15$. Specifically, the figure shows the effects of trade liberalizations on several measures of wage inequality in the calibrated no-free-entry model, decomposing total effects into the contributions of each of the three channels defined in section 5—selection-into-activity, intensive-margin and extensive-margin channels. As discussed earlier, for lower values of ρ, a given change in trade costs leads to larger changes in all measures of wage wage inequality. However, the relative quantitative importance of each channel is not significantly affected by the value of ρ. In particular, the quantitative role of the extensive-margin channel, with drives much of the ambiguity in the theoretical results, is always small. Accordingly, wage inequality increases pervasively following a decline in variable trade costs.

Figure 12: Trade Liberalization and Skill-Productivity Sustituability



Note: The figure illustrates the distributional effects of trade liberalizations in the calibrated no-free-entry model for alternative values of ρ, decomposing total effects into the contributions of each of the three channels defined in section 5—selection-into-activity, intensive-margin and extensive-margin channels. For $\rho = -5$ and $\rho = -15$, panels (a) and (b) show, respectively, the incremental change in the Gini index (black dots) and the contribution of each of these channels (stacked bars) as variable trade costs are incrementally reduced by same proportion $\widehat{\tau}_{step} \approx 0.93$. The horizontal axis indicates the cumulative decline in trade costs after k sequential liberalizations, $\widehat{\tau} = [\widehat{\tau}_{step}]^k$. The rest of the panels show similar calculations for the 90/50 ratio (panels c and d) and the 50/10 ratio (panels e and f).