Trade, Matching and Inequality*

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Abstract

This paper develops a general equilibrium model for examining the effect of trade on the wage distribution that emphasizes within-industry reallocation and heterogeneity of firms and workers. The exit of the least productive firms and the selection into trade of the most productive ones induce labor reallocations in which workers experience "firm" upgrading driving up the high-to-low wage ratio of any pair of workers.

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1 Introduction

The best known general equilibrium model of International Trade, the Heckscher-Ohlin (H-O) model, and its companion theorem, the Stolper-Samuelson Theorem predict that a trade-liberalization-induced increase in the price of unskilled-labor-intensive products should increase the return to the factor that is intensively used in the production of these products, unskilled labor. Based on this theorem and the empirical evidence that developing countries are richly endowed with unskilled labor one would expect the changes in wages induced by trade liberalization to favor the unskilled workers.

However this prediction does not seem to fit the patterns in the data well. Many empirical studies have documented an increase in wage inequality after episodes of trade liberalization in developing countries, that is, the distributional change went in the opposite direction from the one suggested by the H-O model.[1] Moreover, a fundamental prediction of the factor endowment based trade theories is that the adjustment process to trade reforms would involve labor reallocations from sectors that experience a price decline and hence contract, toward sectors that experience relative price increases and hence expand. However, most studies of trade liberalization in developing countries find little evidence in support of such reallocation of resources across sectors. On the other hand, studies that use firm-level data typically find major factor reallocations towards the most productive firms.

The previous facts suggest that within-industry reallocation of resources may be playing a major role in the way trade affects inequality. There is a vast empirical and theoretical economic literature about the relationship between trade and within-sector reallocation of resources. Among the theoretical research, the paper by Melitz in 2003 stands out as one of the most important contributions. Since then, the Melitz (2003) model has become the workhorse framework to analyze the effects of within-sector trade-induced reallocation of resources. In this model firms differ in their productivity and the market structure is that of monopolistic competition. Besides the usual gains from trade derived from more varieties of products available to consumers that is common to all models in which preference for variety motivates trade, this model emphasizes the gains derived from the reallocation of resources toward more productive firms that trade generates: when an economy moves from autarky to trade, the tougher competition makes the least productive firms exit the market while only the most productive ones export. In this paper I refer to these effects as "selection into activity" and "selection into trade", respectively. The exit of the least productive firms and the discrete jump in the demand experienced by exporters generate a reallocation of resources towards more productive firms increasing the overall productivity of the economy.

The message of Melitz (2003) is clear: the selection effects from trade and the consequent within-sector reallocation of resources are beneficial for the economy since they increase overall productivity. But this is only one part of the story. In the original Melitz model the workforce is modeled as a mass of identical individuals making the model silent about the effects of trade on wage

[1] Goldberg-Pavcnik (2004) present an excellent survey of the empirical studies on this topic.

inequality.[2] However in the real world, workers differ in their skill levels and many empirical studies find that more productive firms have a workforce of higher average ability than less productive firms and pay higher wages. When we think about the trade-induced selection effects emphasized by the Melitz (2003) model in this context we immediately see that they should have an asymmetric effect on workers of different ability: i) Selection into activity: the least productive firms exit the market which are the ones employing the least skilled workers; ii) Selection into trade: only the most productive firms export, implying that there is a discrete jump in the demand of high productivity firms which are the ones that employ the high-skill workers. The analysis of impact of these two effects on wage inequality is the central topic of this paper.

To analyze the impact on inequality of the selection effects described above I build a single industry model that features firm and worker heterogeneity in a context of monopolistic competition in the final goods market and competitive factor markets. Specifically, starting from a Melitz (2003) model I introduce a production technology in which high-skill workers have a comparative advantage in production at high productivity firms. This feature of the model gives in equilibrium the positive assortative matching between firms and workers that we see in the data. Two immediate questions arises: First, do the selection mechanisms explained above survive in this new setting? Second, can the trade-induced selection mechanisms alone increase wage inequality?[3] The most important result of this paper is that the answers to these questions are yes: we still have selection into activity and selection into trade and each of these selection mechanisms generates a pervasive increase in wage inequality when moving from autarky to trade.[4] Each of these mechanisms induces a reallocation of resources from low to high productivity firms, and the resulting "firm upgrading" experienced by workers generates a pervasive increase in wage inequality.[5]

The rest of the paper is organized as follows. Section 2 discusses the related literature. Section 3 presents the model. Section 4 characterizes the equilibrium in the closed economy. Section 5 presents the assumptions of the open economy model and characterizes the equilibrium. Section 6 analyses the impact of trade on inequality in an initially autarkic economy. Section 7 concludes. Appendix A and B present Lemmas and proofs of some results in the text.

[2] As was mentioned before, Melitz (2003) is concerned with the effects of trade on the reallocation of resources and the overall productivity of the economy.

[3] There are some models in which the exposure of an economy to trade induces some endogenous actions by firms other than just hiring workers. For instance, trade induces skill-biased technological upgrading in Bustos (2011) or increased screening in a context of imperfect labor markets as in Helpman, Itskhoki and Redding (2010). In my model trade only affects the decision of which kind of worker to hire and how many of them to hire.

[4] By a pervasive increase in inequality I mean that the high-to-low wage ratio of any pair of workers increases. It can be shown that this implies that the wage distribution in autarky Lorenz-dominates the wage distribution corresponding to trade, which in turn means that the wage inequality measured by the Gini Index is higher under trade.

[5] Firms experience "Skill downgrading".

2 Related Literature

As a result of the failure of the traditional neoclassical theory in explaining the empirical patterns mentioned in the introduction, the literature has proposed several channels through which trade liberalization can affect wage inequality in developing countries. This channels include labor market imperfections (Davis and Harrigan (2007); Egger and Kreickmeier (2009); Helpman, Itskhoki and Redding (2010)); endogenous skill-biased technological change (Acemoglu (2003), Bustos (2011)); within industry reallocation of labor towards exporters in models in which the exporting technology is skill intensive.

This paper is part of the rapidly growing literature using assignment models in an international context. Among these papers we can mention Grossman and Maggi (2000), Grossman (2004), Yeaple (2005), Ohnsorge and Trefler (2007), Costinot and Vogel (2010), Monte (2011), for applications to international trade and Kremer and Maskin (2003), Antras, Garicano and Rossi-Hansberg (2006) and Nocke and Yeaple (2008) for applications to offshoring.

While this paper is methodologically close to Costinot and Vogel (2010), the market structure differs substantially. In Costinot and Vogel (2010) all markets are competitive and so the only motivation for trade is comparative advantage. Then trade can only affect inequality if it takes place between countries with different labor endowments. In this paper, the market of final goods is characterized by monopolistic competition and countries trade due to their preference for variety. In contrast to Costinot and Vogel (2010), trade affects wage inequality even when it takes place between identical countries.

This paper is close to Monte (2011) where a traditional Melitz (2003) model is extended along the lines of Lucas (1978): workers are heterogeneous in their managerial skills while they are identical as production workers. In the model managers and "ideas" are combined through a supermodular function to obtain the firm TFP and then production workers are hired as in the traditional Melitz (2003) model. Due to the supermodularity of TFP with respect to ideas and managers, the equilibrium features positive assortative matching between ideas and managers. One important difference with this paper is that in Monte (2011) trade does not induce any change in the matching of managers to ideas for those workers that are still managers after trade is open. In terms of results, the author finds that trade increases inequality at the top of the managers's wage distribution and reduces it at the bottom of the distribution. In this paper inequality increases pervasively as a consequence of trade.

Burstein and Vogel (2012) develop a multi-country, multi-sector quantitative model of trade to asses the effects of trade on inequality. There are only two factors of production, skilled and unskilled labor, and firms are heterogeneous in their productivity and in the degree of the skill-bias of their technology. In particular, more productive firms have a more skill-biased technology. This specification allows trade to affect inequality through two channels: the traditional H-O mechanism that is the result of the reallocation of resources between sectors, and the skill-bias technology mechanism that is the result of the reallocation of resources towards more productive firms within each sector. While the effect of the H-O mechanism on the skill premium depends on the endowment and comparative advantage of each country, the skill-biased technology mechanism increases the skill premium in every country. They calibrate the model to the data and find that the skill premium rises in all countries, providing evidence of the empirical relevance of the second mechanism. The mechanism proposed in this paper is similar in spirit to the skill-biased technology mechanism of Burstein-Vogel (2012) since it focuses on the reallocation of resources toward more productive firms. However, in the present paper the market structure differs substantially and having a continuum of skills potentially allows to analyze the effect of trade on the entire distribution of wages.

At a very advanced stage of this project I became aware of the existence of the working paper version of the now published paper Sampson (2014). In that paper the author analyzes the effect of trade on inequality in an environment very similar to the one proposed here. He develops two models: 1) a stochastic productivity model and 2) a productivity choice model. The stochastic productivity model introduces the strict log-supermodular technology present in Costinot-Vogel (2010) in an otherwise standard Melitz model in a similar way as it is done in the present paper. Even though a number of results are developed in this environment, the main focus of the paper is the productivity choice model. The later model, which extends Yeaple (2005) to a continuum of available technologies, features productivity choice and R&D by firms. In the model, trade affects inequality through its impact on firms's R&D decisions.

It goes without saying that the analysis corresponding to the stochastic productivity case in Sampson (2014) is the closest to the one presented here. However, there are some differences in terms of results due to some differences in the environment. In the present paper I do not include a free entry condition while Sampson (2014) does. While the absence of a free entry condition implies that trade pervasively increases inequality among all workers, Sampson (2014) is able to show through simulations of the model that for some particular assumptions about technology and parameters, inequality may not rise among all workers when the model features a free entry condition.

Some of the main contributions of the present paper are the methods developed for the analysis, since they turn the problem tractable even in the case where only some producers export. These methods allow me to give a full general equilibrium characterization of the model featuring stochastic productivity and show how the selection mechanisms induced by trade affect inequality. These methods can also be applied in a model with a free entry condition and can help understand why this change in the environment introduces the aforementioned ambiguity in terms of the effect of trade on inequality.

3 The Model

3.1 Demand

The preferences of the representative consumer are given by a C.E.S utility function over a continuum of goods indexed by ω :

$$U=\left[\int_{\omega\in\Omega}q\left(\omega\right)^{\frac{\sigma-1}{\sigma}}d\omega\right]^{\frac{\sigma}{\sigma-1}},$$

where the measure of the set Ω represents the mass of available goods and $\sigma>1$ is the elasticity of substitution between goods. The demand and expenditure for individual varieties generated by this utility function are

$$\begin{aligned} q\left(\omega\right) &= RP^{\sigma-1}p\left(\omega\right)^{-\sigma}, \\ r\left(\omega\right) &= RP^{\sigma-1}p\left(\omega\right)^{1-\sigma}, \end{aligned} \tag{1}$$

where P is the aggregate price level and R is the aggregate expenditure,

$$\begin{aligned} P &= \left[\int_{\omega\in\Omega}p\left(\omega\right)^{1-\sigma}d\omega\right]^{\frac{1}{1-\sigma}}, \\ R &= \int_{\omega\in\Omega}r\left(\omega\right)d\omega. \end{aligned} \tag{2}$$

3.2 Production

There is a continuum of active firms in the market, each choosing to produce a different variety ω. In the model, firms and workers are heterogenous; firms differ in their productivity level ϕ and workers differ in their skill level s. The distribution of skills in the economy is represented by the non-negative density $V\left(s\right)$. If $L>0$ denotes the total mass of workers in the economy, then $LV\left(s\right)\geq0$ represents the inelastic supply of workers with skill s. I only consider skill distributions such that the support of V is equal to some bounded interval of non-negative real numbers, i.e. $S\equiv\{s:V\left(s\right)>0\}=[\underline{s},\overline{s}]\subseteq\mathbb{R}_{+}$. In addition, I assume that $V\left(s\right)$ is twice continuously differentiable on S.

The production technology of firms is represented by a cost function that exhibits constant marginal cost and fixed overhead costs. After paying the fixed costs of production described later, a firm must decide how many workers of each skill level to employ. Letting $l\left(s,\phi\right)$ denote the total number of workers of skill s employed by a firm with productivity ϕ, the total output $q\left(\phi\right)$ of the firm is

$$q\left(\phi\right)=\int_{s\in S}A\left(s,\phi\right)l\left(s,\phi\right)ds, \tag{3}$$

where the function $A\left(.,.\right)$ satisfies $A\left(s,\phi\right)>0$, $A_{s}\left(s,\phi\right)>0$ and$A_{\phi}\left(s,\phi\right)>0$.[6] In addition, I

[6] For any function $F(x_1,...,x_n)$, F_{x_i} denotes the partial derivative of F with respect to variable x_i.

assume that $A(.,.)$ is strictly log-supermodular, which by definition implies

$$A\left(s',\phi'\right)A\left(s,\phi\right) > A\left(s',\phi\right)A\left(s,\phi'\right) \text{ for all } s' > s \text{ and } \phi' > \phi. \tag{4}$$

Since $A(s,\phi) > 0$, we can rearrange (4) as $A\left(s',\phi'\right)/A\left(s',\phi\right) > A\left(s,\phi'\right)/A\left(s,\phi\right)$. In other words, the log-supermodularity of $A(.,.)$ implies that high-skill workers have a comparative advantage in production at high productivity firms.

The goal of the paper is to analyze the impact of trade on the wage schedule $w(s)$ through the endogenous decisions of firms regarding the type of workers they hire.[7] In order to isolate this effect, I make assumptions about the nature of fixed costs that guarantee that the fixed-cost-induced demand for labor has no effect on the wage schedule $w(s)$. In particular, I assume that firms pay a fixed cost of $fV(s)$ units of each skill $s \in S$, implying that the total fixed cost of a firm is

$$f\int_{\underline{s}}^{\overline{s}} w(s)\,V(s)\,ds = f\overline{w},$$

where $\overline{w}$ is the average wage which I set as the numeraire, $\overline{w} = 1$. This assumption about fixed costs guarantees that after all firms pay their fixed costs, the distribution of skills in the economy is still given by $V(s)$.

The linear production technology described in (3) implies that the marginal cost of a firm with productivity ϕ, $c(\phi)$, is given by

$$c(\phi) = \min_{s\in S}\left\{\frac{w(s)}{A(s,\phi)}\right\}. \tag{5}$$

This and assumption $A_\phi(s,\phi) > 0$ imply that the marginal cost of a firm decreases with its productivity, a result that I state formally in the next Lemma.

Lemma 1 *For any wage schedule $w(s)$, $c(\phi)$ is strictly decreasing in ϕ.*

Proof. If a firm with productivity ϕ is using a worker with skill s to produce, then a firm with productivity $\phi' > \phi$ can hire a worker of the same skill level and so

$$c\left(\phi'\right) \leq w(s)/A\left(s,\phi'\right) < w(s)/A(s,\phi) = c(\phi),$$

where the strict inequality comes from the assumption $A_\phi(s,\phi) > 0$. ■

The iso-elastic demands given in (1) imply that firms optimally set the price equal to a constant markup over their own marginal costs, $p(\phi) = \frac{\sigma}{\sigma-1}c(\phi)$. This pricing rule together with the cost

[7] $w(s)$ is the wage of a worker with skill level s.

minimization condition (5) imply

$$
\begin{aligned}
p(\phi) &\leq \frac{\sigma}{\sigma-1}\frac{w(s)}{A(s,\phi)} \text{ for all } s \in S \text{ and} \\
p(\phi) &= \frac{\sigma}{\sigma-1}\frac{w(s)}{A(s,\phi)} \text{ if } l(s,\phi) > 0.
\end{aligned}
\tag{6}
$$

Finally, the pricing rule described above and (1) imply that a firm's demand, revenue and profit are given by

$$
\begin{aligned}
q(\phi) &= RP^{\sigma-1}\left[\frac{\sigma}{\sigma-1}c(\phi)\right]^{-\sigma}, \\
r(\phi) &= RP^{\sigma-1}\left[\frac{\sigma}{\sigma-1}c(\phi)\right]^{1-\sigma}, \\
\pi(\phi) &= r(\phi) - c(\phi)\,q(\phi) - f = \frac{r(\phi)}{\sigma} - f.
\end{aligned}
\tag{7}
$$

4 Equilibrium in the Closed Economy

4.1 Definition of Equilibrium

There is a fixed mass $\overline{M}$ of potential firms in the industry. The productivity distribution of these potential firms is given by $G(\phi)$ with density function $g(\phi)$. I only consider distributions such that the support of g, $\Phi \equiv \{\phi : g(\phi) > 0\}$, is equal to some bounded interval of non-negative real numbers, $[\underline{\phi}, \overline{\phi}] \subseteq \mathbb{R}_+$.

A firm is active in the market if and only if it makes positive profits. Since Lemma 1 and the expressions in (7) imply that the profit of a firm is increasing in the firm's productivity, then there is a cutoff productivity value ϕ^* such that only firms with productivity above this value are active in the market. This cutoff value corresponds to the level of productivity at which firms make zero profits,

$$\pi(\phi^*) = 0. \tag{8}$$

In turn, the cutoff ϕ^* determines the total mass of active firms in the industry,

$$M = [1 - G(\phi^*)]\,\overline{M}. \tag{9}$$

Finally, the labor market clearing condition requires that the total supply of workers with skill level s equals its demand,

$$
\begin{aligned}
LV(s) &= \int_{\phi^*}^{\overline{\phi}} l(s,\phi) \frac{g(\phi)}{[1-G(\phi^*)]} d\phi M + MfV(s) \\
&= \int_{\phi^*}^{\overline{\phi}} l(s,\phi) g(\phi) d\phi \overline{M} + MfV(s),
\end{aligned} \tag{10}
$$

where the first and second term in the right hand side of the last expression reflect the demand of labor derived from variable production and fixed costs, respectively.

Having described all the components of the economy, we can now state a formal definition of the equilibrium.

Definition 1 *An autarky equilibrium is a mass of active firms $M > 0$, a cutoff value ϕ^*, and a set of functions $q : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, $l : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $w : S \to \mathbb{R}_+$ such that conditions (3),(6)-(10) hold.*

4.2 Characterization of the Equilibrium

The log-supermodularity of $A(.,.)$ has strong implications for the equilibrium allocations. As discussed above, this assumption implies that high-skill workers have a comparative advantage in production at high productivity firms. As a consequence of this property, the equilibrium labor allocation is characterized by positive assortative matching, i.e. more productive firms employ workers with higher skill levels. Moreover, each firm employs workers of a single skill level and all workers with the same skill level are employed at firms with the same productivity level. This property of the equilibrium is formally stated in the following Lemma.

Lemma 2 *In an autarky equilibrium, there exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$ such that (i) $l(s,\phi) > 0$ if and only if $N(s) = \phi$, (ii) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$.*

Proof: See Appendix B.

Even though the proof of this Lemma is somewhat technical, the key mechanism driving the positive assortative matching is very intuitive. Suppose that in equilibrium, a firm with productivity ϕ is employing a worker with skill s. Cost minimization implies that the marginal cost of production that the firm achieves with that worker is less or equal than the one it would obtain from employing a worker of skill $s' \neq s$, $w(s)/A(s,\phi) \leq w(s')/A(s',\phi)$. If $s < s'$, then the last result implies that for any firm with productivity $\phi' < \phi$ we have

$$
\frac{A(s',\phi')}{A(s,\phi')} < \frac{A(s',\phi)}{A(s,\phi)} \leq \frac{w(s')}{w(s)},
$$

where the first inequality is obtained using the strict log-supermodularity of $A(.,.)$ and the second one reflects the implications of cost minimization discussed before. Notice that the previous chain of inequalities implies that a firm with productivity ϕ' will never hire a worker with skill s' since it can obtain a strictly lower marginal cost hiring a worker with skill s. Then any worker with skill $s' > s$ must be employed at a firm with productivity greater than or equal to ϕ. Although this argument only proves that the matching function is weakly increasing, it shows the main mechanism driving the positive assortative matching.

The previous Lemma allows me to recast the equilibrium conditions in terms of the matching function N. In fact, those equilibrium conditions yield a system of differential equations from which the cutoff productivity value ϕ^*, the wage schedule $w(s)$, the price function $p(\phi)$ and the matching function N can be jointly solved. Letting H denote the inverse function of the matching function N, this result is stated in the following Lemma.

Lemma 3 *In an autarky equilibrium with exit cutoff* ϕ^*, *the wage schedule* $w : S \to \mathbb{R}_+$ *and the price function* $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ *are continuously differentiable functions; the matching function* $N : S \to [\phi^*, \overline{\phi}]$ *is twice differentiable and they all satisfy*

$$\frac{d \ln w(s)}{ds} = \frac{\partial \ln A(s, N(s))}{\partial s}, \tag{11}$$

$$\frac{d \ln p(\phi)}{d\phi} = -\frac{\partial \ln A(H(\phi), \phi)}{\partial \phi}, \tag{12}$$

$$\frac{A(s, N(s))[L - fM]V(s)}{g(N(s))\overline{M}N_s(s)} = QP^{\sigma} p(N(s))^{-\sigma}, \tag{13}$$

$$QP^{\sigma} p(N(\underline{s}))^{1-\sigma} = \sigma f, \tag{14}$$

with $N(\underline{s}) = \phi^*$, $N(\overline{s}) = \overline{\phi}$, *and where* Q *denotes the consumption aggregator and* $p(N(s))$ *is given by (6).*

Proof: See Appendix B.

While the formal proof of the previous Lemma can be found in the appendix, here I provide some intuition for the results assuming away the differentiability of the endogenous functions involved.

Except for condition (13), the interpretation of the conditions in the last Lemma is straightforward. Condition (11) says that if a worker with skill s is assigned to a firm with productivity $N(s)$ in equilibrium, then the skill s must satisfy the first-order condition of the cost minimization problem of a firm with that productivity. Condition (12) is obtained log-differentiating the equilibrium pricing rule $p(\phi) = \frac{\sigma}{\sigma-1}\frac{w(H(\phi))}{A(H(\phi),\phi)}$ with respect to ϕ and using (11).[8] Condition (14) ensures that firms with cutoff productivity $\phi^* = N(\underline{s})$ make zero profits.

Condition (13) deserves more explanation since it involves the slope of the matching function. The condition states that in equilibrium the supply of each variety (left hand side) must be equal to its demand (right hand side). As we can see, the total supply of variety $N(s)$ depends negatively

[8] Notice that this way of obtaining (12) is using the differentiability of the matching function. However, in the appendix it is shown that this condition can be derived without making use of the differentiability of N.

on the slope of the matching function at s, $N_s(s)$. To understand the intuition behind this result suppose that we have two matching functions N,N' such that $N(s) = N'(s)$ for some $s \in S$ but $N_s(s) > N'_s(s)$. Then the skills in the interval $[s, s+ds]$ are assigned to firms with productivity in the intervals $[N(s), N(s) + N_s(s)\,ds]$ and $[N(s), N(s) + N'_s(s)\,ds]$, respectively. Then if $N_s(s) > N'_s(s)$ this means that the skills $[s, s+ds]$ are assigned to more firms under N than under N'. Then the production of the firms with productivity in $[N(s), N(s) + N_s(s)\,ds]$ must be smaller under N than under N', and this is exactly what the left hand side of (13) shows.

Condition (11) will be crucial in the analysis of the effect on inequality of changes in the environment. It establishes a direct link between changes in wage inequality and shifts of the matching function.[9] This can be seen more clearly looking at the wage ratio corresponding to two different skill levels s and s''. Assuming $s'' > s'$ and integrating (11) between s' and s'' we get

$$\frac{w(s'')}{w(s')} = \exp\left\{\int_{s'}^{s''} \frac{\partial \ln A(t, N(t))}{\partial s} dt\right\}. \tag{15}$$

Due to the strict log-supermodularity of $A(.,.)$ the ratio $w(s'')/w(s')$ is increasing in the values that the matching function takes on $[s', s'']$. Then any change in the environment that shifts up the matching function on that interval increase the high-to-low-skill wage ratios in the interval. It can be shown that when this is the case, the new distribution of wages on the interval $[s', s'']$ will be second-order stochastically dominated by the previous one, that is, inequality is pervasively higher after the change.[10]

Lemma 3 also allows us to characterize the evolution of output and revenues as we move along the productivity level. The CES demand structure and equation (12) imply that the ratio of any two firms' outputs and revenues depends only on their productivity levels and the matching function between these two levels,

$$\frac{q(\phi'')}{q(\phi')} = \left[\frac{p(\phi'')}{p(\phi')}\right]^{-\sigma} = \exp\left\{\sigma \int_{\phi'}^{\phi''} \frac{\partial \ln A(H(t), t)}{\partial \phi} dt\right\}, \tag{16}$$

$$\frac{r(\phi'')}{r(\phi')} = \left[\frac{p(\phi'')}{p(\phi')}\right]^{1-\sigma} = \exp\left\{(\sigma - 1) \int_{\phi'}^{\phi''} \frac{\partial \ln A(H(t), t)}{\partial \phi} dt\right\}. \tag{17}$$

The log-supermodularity of $A(.,.)$ implies that the ratio of revenues $r(\phi'')/r(\phi')$ is increasing in the values that the *inverse* of the matching function takes on $[\phi', \phi'']$. Notice that this means that shifts in the matching function will have opposite effects on the dispersion of wages and revenues: an upward shift of the matching function increases increase the dispersion of wages at the same time it reduces the dispersion in revenues.[11]

[9] This feature of the model is also present in Costinot-Vogel (2010).

[10] In Appendix B I show that the new distribution is Lorenz dominated by the previous one. The equivalence between Lorenz dominance and normalized second-order stochastic dominance was first shown in Atkinson (1970).

[11] The inverse of the matching function, H, shifts down.

The characterization of the autarky equilibrium given in Lemma 3 is in terms of a system of non-linear differential equations which is a mathematical object that is usually difficult to work with. The following Lemma presents a characterization of the equilibrium exit cutoff and the matching function in terms of a second-order differential equation that does not involve any other endogenous variables.

Lemma 4 *A function $N : S \to [\phi^*, \overline{\phi}]$ and a number ϕ^* are the matching function and the exit cutoff corresponding to an autarky equilibrium if and only if the following conditions hold.*
(i) Given ϕ^*, the matching function N satisfies the second order ordinary differential equation[12]

$$\begin{aligned} N_{ss}(s) &= \left[\frac{A_s(s, N(s))}{A(s, N(s))} + \frac{V_s(s)}{V(s)}\right] N_s(s) + \cdots \\ &- \left[(\sigma - 1)\frac{A_\phi(s, N(s))}{A(s, N(s))} + \frac{g_\phi(N(s))}{g(N(s))}\right] N_s(s)^2, \end{aligned} \tag{18}$$

with boundary conditions $N(\underline{s}) = \phi^*$ and $N(\overline{s}) = \overline{\phi}$.
(ii) The exit cutoff ϕ^* and the matching function N satisfy

$$\overline{M}(\sigma - 1) f \int_{\phi^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma - 1}\right] g(\phi)\, d\phi = L, \tag{19}$$

where $H(t) \equiv N(t)^{-1}$. Moreover, if the mass of potential firms is sufficiently high, then there exists a unique autarky equilibrium.

Proof: See Appendix B.

The "only if" part of the lemma is obtained as follows. Log-differentiating both sides of (13) with respect to s and using (12) yields (18). In Appendix B, I show that equation (19) guarantees that the total value of wages that firms pay to the workers employed in variable production equals the wages that these workers get when the firms with productivity ϕ^* are making zero profits. To prove the "if" part of the Lemma, in Appendix B I show how quantities, prices and wages satisfying (11)-(14) can be constructed from an exit cutoff and a matching function that satisfy (18) and (19).

To gain more insight into the previous Lemma, I will elaborate more on the intuition behind it. Let $N(s)$ and $w(s)$ be a matching function and a wage schedule satisfying conditions (11) and (13). Then, for any positive scalar K, $N(s)$ and $Kw(s)$ also satisfy the same conditions. This means that the wage schedule affects the matching function only through its growth, and not through its level.[13] But in turn, the growth of the wage schedule depends only on the matching function. For this reason, I can arrive to a characterization of the matching function that does not include any endogenous variables other than the exit cutoff.

[12] Appendix A presents some results concerning this second-order differential that will be repeatedly used in the text and in Appendix B.

[13] See condition (11).

To conclude this section I summarize the qualitative properties of the equilibrium. As in Melitz (2003), more productive firms generate more revenue and produce more output. The distinctive feature of this model is that the equilibrium is characterized by positive assortative matching between firms and workers: more productive firms employ workers of higher ability. This last fact immediately implies that more productive firms pay higher wages.[14] All these features of the model fit well the patterns that we observe in the data.

5 Equilibrium in the Open Economy

5.1 Definition of Equilibrium

Trade takes place between two symmetric (identical) economies of the type described before. A firm that wishes to export has to pay a fixed cost of $f_x V(s)$ units of each skill $s \in S$.[15] There are also per-unit trade costs which are modeled in the standard iceberg formulation, whereby $\tau > 1$ units of a good must be shipped in order for 1 unit to arrive in a foreign destination. The symmetry assumption ensures that both countries share the same equilibrium variables including the wage and price schedules. Consequently, it is enough to analyze the equilibrium in one country. As in the previous section, I set the common average wage $\overline{w}$ as the numeraire. In what follows I use a subscript d to refer to a variable in the domestic market and a subscript x for variables in the foreign market with the only exception being the domestic price schedule which does not have any subscript.

As in the closed economy, a firm sets the price in the domestic market, $p(\phi)$, according to (6). A firm that exports sets a higher price in the foreign market reflecting the increased marginal cost of serving that market, $p_x(\phi) = \tau p(\phi)$. Then, a firm's demand, revenue and profit in the domestic and foreign markets are

$$
\begin{aligned}
q_d(\phi) &= RP^{\sigma-1}\left[\frac{\sigma}{\sigma-1}c(\phi)\right]^{-\sigma}, & q_x(\phi) &= \tau^{-\sigma}q_d(\phi), \\
r_d(\phi) &= RP^{\sigma-1}\left[\frac{\sigma}{\sigma-1}c(\phi)\right]^{1-\sigma}, & r_x(\phi) &= \tau^{1-\sigma}r_d(\phi), \\
\pi_d(\phi) &= \frac{r_d(\phi)}{\sigma} - f, & \pi_x(\phi) &= \frac{r_x(\phi)}{\sigma} - f_x.
\end{aligned} \tag{20}
$$

[14] Wages are strictly increasing in the skill of the workers.

[15] As in the case of the fixed costs of production, this assumption guarantees that the demand of labor induced by the fixed exporting costs does not affect the equilibrium wage schedule $w(s)$.

The last expressions imply total output and total revenue of a firm with productivity ϕ are[16]

$$
\begin{aligned}
q(\phi) &= \begin{cases} q_d(\phi) & \text{if the firm does not export} \\ \left(1+\tau^{1-\sigma}\right) q_d(\phi) & \text{if the firm exports} \end{cases} \\
r(\phi) &= \begin{cases} r_d(\phi) & \text{if the firm does not export} \\ \left(1+\tau^{1-\sigma}\right) r_d(\phi) & \text{if the firm exports} \end{cases}
\end{aligned} \tag{21}
$$

There is a fixed mass $\overline{M}$ of potential firms in the industry. The productivity distribution among these firms is given by $G(\phi)$. Lemma 1 and the expressions in (20) imply that the profits that a firm makes in the domestic market and in the foreign market are increasing in the firm's productivity. Consequently, there are cutoff productivity values ϕ^* and ϕ_x^* such that a firm with productivity ϕ produces if and only if $\phi \in [\phi^*, \overline{\phi}]$ and exports if and only if $\phi \in [\phi_x^*, \overline{\phi}]$. These cutoffs are determined from the following zero-profit conditions

$$
\begin{aligned}
\pi_d(\phi^*) &= 0, \\
\pi_x(\phi_x^*) &= 0.
\end{aligned} \tag{22}
$$

In turn, the cutoff ϕ^* determines the total mass of active firms in the industry and the cutoff ϕ_x^* determines the mass of exporting firms

$$
\begin{aligned}
M &= [1 - G(\phi^*)]\,\overline{M}, \\
M_x &= [1 - G(\phi_x^*)]\,\overline{M}.
\end{aligned} \tag{23}
$$

I assume $\tau^{\sigma-1} f_x > f$, which guarantees that the equilibrium features the selection of more productive firms into export markets, $\phi_x^* > \phi^*$.

Finally, the labor market clearing condition requires that the total supply of workers with skill level s equals its demand,

$$
LV(s) = \int_{\phi^*}^{\overline{\phi}} l_d(s,\phi)\, g(\phi)\, d\phi \overline{M} + \int_{\phi_x^*}^{\overline{\phi}} l_x(s,\phi)\, g(\phi)\, d\phi \overline{M} + fMV(s) + f_x M_x V(s), \tag{24}
$$

where $l_d(s,\phi)$ and $l_x(s,\phi)$ represent the labor used in the production of goods sold domestically and exported, respectively.

Having described all the components of the open economy, we can now state a formal definition of the equilibrium.

Definition 2 *A trade equilibrium is a mass of active firms $M > 0$, a mass of exporting firms $M_x > 0$, cutoff values ϕ^* and ϕ_x^*, and a set of functions $q, q_d : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, $l : S \times [\phi^*, \overline{\phi}] \to \mathbb{R}_+$, $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ and $w : S \to \mathbb{R}_+$ such that conditions (3),(6),(20)-(24) hold.*

[16] Total output includes the inceberg trade costs.

5.2 Characterization of the Equilibrium

As in the closed economy, cost minimization and the strict log-supermodularity of $A(.,.)$ imply that the equilibrium labor allocation is characterized by positive assortative matching, a result that is stated in the next Lemma.

Lemma 5 *In a trade equilibrium, there exists a continuous and strictly increasing matching function $N : S \to [\phi^*, \overline{\phi}]$ such that (i) $l(s,\phi) > 0$ if and only if $N(s) = \phi$, (ii) $N(\underline{s}) = \phi^*$, and $N(\overline{s}) = \overline{\phi}$.*

Proof: Same proof as in Lemma 2.

The assumption on trade costs we made above, $\tau^{\sigma-1} f_x > f$, implies that the trade equilibrium is characterized by selection into export markets of the most productive firms, i.e. the export productivity cutoff is strictly higher than the exit productivity cutoff, $\phi_x^* > \phi^*$. This fact together with the previous Lemma implies the there is a cutoff skill level $s_x^* = H(\phi_x^*)$ such that all workers with skill level above s_x^* are employed by firms that export. The selection into trade introduces a discrete upward jump in the size[17] of firms at the cutoff value ϕ_x^*, which implies that now the matching function presents a kink at the skill level s_x^*. The following Lemma formalizes the previous argument and gives a characterization of the equilibrium in the open economy analogous to Lemma 3.

Lemma 6 *Let ϕ^*, ϕ_x^* and s_x^* be the cutoffs described above. Then, in a trade equilibrium with cutoffs ϕ^*, ϕ_x^*, s_x^*, the wage schedule $w : S \to \mathbb{R}_+$ and the price function $p : [\phi^*, \overline{\phi}] \to \mathbb{R}_+$ are continuously differentiable functions; together with the matching function $N : S \to [\phi^*, \overline{\phi}]$ they all satisfy*

$$\frac{d \ln w(s)}{ds} = \frac{\partial \ln A(s, N(s))}{\partial s}, \tag{25}$$

$$\frac{d \ln p(\phi)}{d\phi} = -\frac{\partial \ln A(H(\phi), \phi)}{\partial \phi}, \tag{26}$$

$$N = N^d 1_{[\underline{s}, s_x^*)} + N^x 1_{[s_x^*, \overline{s}]}, \tag{27}$$

where N^d, N^x are twice differentiable functions on $[\underline{s}, s_x^*)$ and $[s_x^*, \overline{s}]$ respectively and satisfy

$$\frac{A\left(s, N^d(s)\right)[L - fM - f_x M_x] V(s)}{g\left(N^d(s)\right) \overline{M} N_s^d(s)} = Q P^\sigma p \left(N^d(s)\right)^{-\sigma} \tag{28}$$

$$\text{with } N^d(\underline{s}) = \phi^*, N^d(s_x^*) = \phi_x^*,$$

$$\frac{A(s, N^x(s))[L - fM - f_x M_x] V(s)}{g(N^x(s)) \overline{M} N_s^x(s)} = \left(1 + \tau^{1-\sigma}\right) Q P^\sigma p \left(N^x(s)\right)^{-\sigma} \tag{29}$$

$$\text{with } N^x(s_x^*) = \phi_x^*, N^x(\overline{s}) = \overline{\phi},$$

[17] By size I mean the number of workers employed at each firm.

$$QP^{\sigma}p\left(N\left(\underline{s}\right)\right)^{1-\sigma}=\sigma f, \tag{30}$$

$$QP^{\sigma}p\left(N\left(s_x^*\right)\right)^{1-\sigma}=\tau^{\sigma-1}\sigma f_x, \tag{31}$$

where M, M_x are given by (23) and Q is the consumption aggregator.

A formal derivation of the previous Lemma is omitted because it follows closely that of Lemma 3. Instead, I present a rather informal analysis of the Lemma emphasizing the intuition leading to each condition.

Conditions (25), (26) and (30) are also present in Lemma 3 and they have the same interpretation here as they had then. They are derived from cost minimization, optimal pricing and the zero domestic profits condition for firms with productivity ϕ^*, respectively. In addition, condition (31) ensures that firms with productivity $\phi_x^* = N\left(s_x^*\right)$ make zero profits from selling in the foreign market.

Conditions (27), (28) and (29) are the open economy version of condition (13). The new conditions reflect the selection into trade of the most productive firms discussed above. In particular, equation (28) states that in equilibrium, the supply of each non-exporting firm (left hand side) must be equal to its demand (right hand side). Similarly, equation (29) states that the total output produced by each exporter must be equal to its demand. Selection into export markets generates a discontinuous upward jump on the demand for varieties at ϕ_x^*, which is reflected in the augmenting factor $\left(1+\tau^{1-\sigma}\right)$ in the right hand side of (29). Notice that the continuity of $A\left(s,\phi\right)$, $V\left(s\right)$, $g\left(\phi\right)$ and $N\left(s\right)$ together with conditions (28) and (29) imply that the matching function presents a concave kink at s_x^*, i.e. $N_s^x\left(s_x^*\right) < N_s^d\left(s_x^*\right)$.

As in the case of Lemma 3, the characterization of the equilibrium given in the last Lemma is in terms of a system of non-linear differential equations which is a complicated mathematical object. The next Lemma is the open economy version of Lemma 4. It presents a characterization of the productivity cutoff values and the piecewise differentiable matching function in terms of second-order differential equations that do not involve any other endogenous variables.

Lemma 7 *A function $N : S \to [\phi^*, \overline{\phi}]$ and numbers ϕ^*, ϕ_x^*, s_x^* are the matching function and cutoffs corresponding to a trade equilibrium of the economy described in Lemma 6 if and only if the following conditions hold.*
(i) N satisfies (27)

$$N = N^d 1_{[\underline{s},s_x^*)} + N^x 1_{[s_x^*,\overline{s}]}.$$

(ii) Given the cutoffs ϕ^*, ϕ_x^*, s_x^*, the functions N^d, N^x satisfy the second-order differential equation (18) with boundary conditions $N^d\left(\underline{s}\right)=\phi^*$, $N^d\left(s_x^*\right)=\phi_x^*=N^x\left(s_x^*\right)$, and $N^x\left(\overline{s}\right)=\overline{\phi}$.
(iii) The right and left derivatives of N at s_x^* satisfy

$$\frac{N_s^x\left(s_x^*\right)}{N_s^d\left(s_x^*\right)}=\frac{1}{1+\tau^{1-\sigma}}<1. \tag{32}$$

(iv) The cutoffs and the matching function satisfy

$$e^{(\sigma-1)\int_{\phi^*}^{\phi_x^*} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} = \tau^{\sigma-1} \frac{f_x}{f} \tag{33}$$

and

$$\overline{M}\left(\sigma-1\right) f \int_{\phi^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g\left(\phi\right) d\phi + \cdots \tag{34}$$
$$\cdots + \overline{M}\left(\sigma-1\right) f_x \int_{\phi_x^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi_x^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g\left(\phi\right) d\phi = L.$$

Moreover, this implies that if $\tau^{\sigma-1} f_x/f$ is not too high, a trade equilibrium exist and it is unique.

Proof: See Appendix B for existence and uniqueness.

The derivation of the previous Lemma follows closely that of Lemma 4. As discussed above, the selection into trade of the most productive firms introduces a kink in the matching function at s_x^*, but it remains twice differentiable on the intervals $[\underline{s}, s_x^*)$, $(s_x^*, \overline{s}]$. This is captured in conditions (i) and (ii). In addition, the continuity of $A\left(s,\phi\right), V\left(s\right), g\left(\phi\right)$ and $N\left(s\right)$ and conditions (28), (29) imply that, at the kink, the right and left derivatives of the matching function satisfy condition (iii).

Condition (iv) reflects some restrictions that the productivity cutoffs and the matching function must jointly satisfy in equilibrium. First, the domestic revenues of export-cutoff firms and exit-cutoff firms are linked in equilibrium. Dividing each side of (31) by the corresponding side of (30) yields $r_d\left(\phi_x^*\right)/r_d\left(\phi^*\right) = \tau^{\sigma-1} f_x/f$, and combining this expression with (17) yields (33). Second, and similar to equation (19) in the closed economy, condition (34) guarantees that the value of the wages paid by firms to the workers used in variable production is equal to the wages that these workers get when the firms with productivity ϕ^* are making zero profits and firms with productivity ϕ_x^* are making zero export-profits.

To conclude this section I summarize the qualitative properties of the equilibrium in the open economy. As in the autarky equilibrium, more productive firms generate more revenue and produce more output. The equilibrium is characterized by positive assortative matching between firms and workers: more productive firms employ workers of higher ability. This last fact immediately implies that more productive firms pay higher wages. When we combine these features with the selection into trade of the most productive firms we get that exporters are bigger in terms of revenue and output, have a workforce of higher average ability and pay higher wages. Again, all these features of the model fit well the patterns that we observe in the data.

6 The Impact of Trade on Inequality

In this section I analyze the impact of trade on an initially autarkic economy. Specifically, I am interested in the questions raised in the introduction: does the model feature selection into activity and selection into trade as in Melitz (2003)? What is the impact of these selection effects on wage inequality? In order to answer these questions it is sufficient to know the effect of trade on the matching function. Therefore, I start this section with the most important result of this paper, Theorem 1, which characterizes the impact of trade on the matching function. In what follows I use a subscript or superscript a to identify the variables that correspond to the autarky equilibrium.

Theorem 1 *Let N^a and N^t be the matching functions that correspond to an autarky and a trade equilibrium, respectively. Then $N^t(s) > N^a(s)$ for all $s \in [\underline{s}, \overline{s})$.*

Proof: See Appendix B.

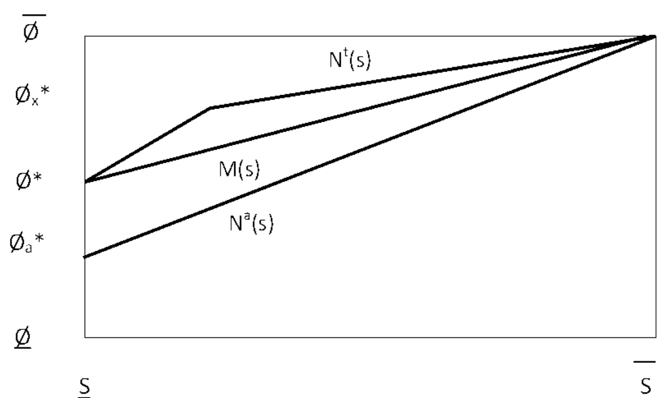


Figure 1

Typical autarky and trade equilibrium matching functions N^a and N^t are represented in Figure 1. Theorem 1 immediately implies that the exposure to trade induces an increase in the exit productivity cutoff $\phi^* > \phi_a^*$. The least productive firms with productivity levels between ϕ_a^* and ϕ^* can no longer earn positive profits and therefore exit. In addition, only the most productive producers find it profitable to pay the fixed cost to export. As in Melitz (2003), both selection effects reallocate market shares toward more productive firms and contribute to an aggregate productivity gain.

Let us turn to the central issue of this paper which is the analysis of the effects of trade on wage inequality. As discussed in Section 4, condition (11) —present in the autarky and trade equilibrium— establishes a direct link between the matching function and the dispersion of the wage schedule.[18] An upward shift in the matching function is associated with higher high-to-low-

[18]It corresponds to Condition (25) in the trade equilibrium.

skill wage ratios. Then an immediate corollary of Theorem 1 is that trade induces a pervasive increase in wage inequality, that is,

$$\frac{w^a(s'')}{w^a(s')} < \frac{w^t(s'')}{w^t(s')} \text{ for all } s'' > s' \in [\underline{s}, \overline{s}].$$

We can decompose the overall increase in wage inequality in two components. The first component reflects the increase in wage inequality due to the selection into activity effect. In Figure 1 the function M represents the matching function corresponding to an economy with exit productivity cutoff ϕ^* and no selection into trade.[19] The change from N^a to M isolates the impact of the selection into activity effect on the matching function. An immediate corollary of Lemma 8.i in Appendix A is that $M(s) > N^a(s)$ for all $s \in [\underline{s}, \overline{s})$. Thus, the selection into activity mechanism alone induces a pervasive increase in wage inequality. The intuition of this result is the following: given the autarky wage schedule, the exit of the least productive firms reduces the relative demand of the least-skilled workers in the economy and so the wages of these workers must decrease relative to the wages of higher-skilled workers to clear the labor market.

The second component captures the increase in wage inequality due to the selection into trade effect. The change from M to N^t reflects the impact of the selection into trade mechanism on the matching function. Notice that M and N^t satisfy the conditions of Lemma 10 in Appendix A and so $N^t(s) > M(s)$ for all $s \in (\underline{s}, \overline{s})$. As before, the selection into trade effect alone induces a pervasive increase in wage inequality. The discrete jump in demand for varieties at ϕ_x^* increases the relative demand of high-skill workers for a given wage schedule. Then wages have to adjust to clear the labor market and the relative wage of high-skill workers increases.

It is important to note that a positive fixed exporting cost is a necessary condition for any of the two trade selection mechanisms discussed above to take place. If we set $f_x = 0$, then the matching function is not affected by trade which means that trade has no effect on inequality. It cannot be overemphasized that when $f_x > 0$, each of the two selection effects induced by trade generates a pervasive increase in inequality. Consequently, even if the values of parameters are such that there is no "selection into trade" effect ($\tau (f_x/f)^{\frac{1}{\sigma-1}} = 1$), inequality in the open economy is still pervasively higher than in autarky due to the "selection into activity" effect.[20] An immediate corollary of the analysis in this section is that as long as the selection into activity effect is present ($\phi^* > \phi_a^*$), then inequality will be pervasively higher in the trade equilibrium.

[19] Notice that once the cutoff is determined, the matching fuction is uniquely determined through the second-order differential equation (4).

[20] This contrasts with Helpman, Itshoki and Redding (2010) where the selection into trade effect is the driving force behind the changes in wage inequality.

7 Conclusion

In this paper I build a general equilibrium model that features firm and worker heterogeneity in a context of monopolistic competition to analyze the effects of trade on wage inequality. To that end I develop tools that allow me to fully solve the model and conduct comparative statics exercises in a very tractable way. The results of this paper suggest that, without resorting to labor market imperfections or skill-bias technological change, the intra-industry reallocation of resources induced by trade alone can have an important impact on inequality.

An important immediate extension of this model would be the introduction of a free entry condition as in Melitz (2003). The tools put forward in this paper can also be used to analyze the impact of trade in that case. An immediate corollary of the analysis in Section 6 is that as long as we have selection into activity, inequality will be pervasively higher in the trade equilibrium. Then the question reduces to analyze under what conditions the selection into activity effect will be present in an extended model that includes a free entry condition.

8 Appendix A

The purpose of this Appendix is to present some results concerning the second-order differential equation (18) that are used repeatedly in the text and in Appendix B.

Lemma 8 (No crossing) *Assume that the conditions*[21] *for existence and uniqueness of a boundary problem including the second-order differential equation (18) are satisfied. Let* $N, N' : [\underline{s}, \overline{s}] \to \mathbb{R}$ *be two functions that satisfy the second-order differential equation (18) with boundary conditions* $N(\underline{s}) = a$, $N(\overline{s}) = b$ *and* $N'(\underline{s}) = a'$, $N'(\overline{s}) = b'$.
(i) if $a' > a$ *and* $b' = b$ *then* $N'(s) > N(s)$ *for all* $s \in [\underline{s}, \overline{s})$ *and* $N'_s(s) < N_s(s)$ *for all* $s \in [\underline{s}, \overline{s}]$.
(ii) if $a' = a$ *and* $b' > b$ *then* $N'(s) > N(s)$ *for all* $s \in (\underline{s}, \overline{s}]$ *and* $N'_s(s) > N_s(s)$ *for all* $s \in [\underline{s}, \overline{s}]$.
(iii) if $a' = a$ *and* $b' = b$ *then* $N'(s) = N(s)$ *for all* $s \in [\underline{s}, \overline{s}]$ *and* $N'_s(s) = N_s(s)$ *for all* $s \in [\underline{s}, \overline{s}]$.

Proof:. (i) Remember that a whole family of functions will satisfy a second-order differential equation like (18). In fact, if the conditions for the existence and uniqueness of a solution to a boundary problem including (18) are satisfied, then to pin down a particular solution is sufficient to know the value of the function or its derivative at two points $s_1, s_2 \in [\underline{s}, \overline{s}]$. This implies that N and N' cannot intersect twice on $[\underline{s}, \overline{s}]$ since by assumption $N \neq N'$ (different boundary conditions). Given that they intersect at $\overline{s}$, and that $N'(\underline{s}) > N(\underline{s})$, then the continuity of N, N' implies $N'(s) > N(s)$ for all $s \in [\underline{s}, \overline{s})$. For the same reasons explained above, the derivative functions N'_s,N_s cannot intersect. Then, this last fact, $N(\overline{s}) = N'(\overline{s})$, $N'(s) > N(s)$ for all $s \in [\underline{s}, \overline{s})$ and the continuity of N'_s and N_s imply $N'_s(s) < N_s(s)$ for all $s \in [\underline{s}, \overline{s}]$.

(ii) The proof follows a similar argument here.

(iii) In this case the functions cross twice on the interval $[\underline{s}, \overline{s}]$ and because of the uniqueness of the solution to (18) it must be the case that N and N' are the same function. ■

Now consider a function N that can be written as follows:

$$N(s) = N^1(s)\, 1_{[\underline{s}, s^*)} + N^2(s)\, 1_{[s^*, \overline{s}]} \tag{35}$$

where the functions N_1 and N_2 satisfy (*18*) with boundary conditions $N^1(\underline{s}) = \phi_0$, $N^1(s^*) = \phi_1 = N^2(s^*)$ and $N^2(\overline{s}) = \phi_2$. The previous function presents some features that are important for our purposes. First, the function N is continuous since $N^1(s^*) = \phi_1 = N^2(s^*)$. Second, once we know the boundary conditions s^*, ϕ_0, ϕ_1 and ϕ_2, the function N is uniquely determined. This is a consequence of the uniqueness of the solution to the second-order differential equation (*18*). Finally, the function N generally presents a kink at s^* and the ratio of right to left derivative $N^2_s(s^*)/N^1_s(s^*)$ is a function of the boundary conditions s^*, ϕ_0, ϕ_1, and ϕ_2. The following Lemma characterizes the behavior of the ratio $N^2_s(s^*)/N(s^*)$ as a function of s^*, ϕ_0, ϕ_1 for a given final condition ϕ_2.

[21] The conditions required for existence and uniqueness are met in the present context. These conditions can be found in Agarwal and Regan (2008). Because of space constraints I have decided not to include those conditions here.

Lemma 9 *Let N^1 and N^2 satisfy the second-order differential equation (18) with boundary conditions $N^1(\underline{s}) = \phi_0$, $N^1(s^*) = N^2(s^*) = \phi_1$ and $N^2(\overline{s}) = \phi_2$. Then*
(i) the ratio $N_s^2(s^*)/N_s^1(s^*)$ is a continuous function of ϕ_0, ϕ_1 and s^*.
(ii) $N_s^2(s^*)/N_s^1(s^*)$ is decreasing in ϕ_1, and increasing in ϕ_0 and s^*.
(iii) $\lim_{\phi_1 \to \phi_2} N_s^2(s^*)/N_s^1(s^*) = 0$, $\lim_{\phi_1 \to \phi_0} N_s^2(s^*)/N_s^1(s^*) = \infty$
(iv) $\lim_{s^* \to \overline{s}} N_s^2(s^*)/N_s^1(s^*) = \infty$, $\lim_{s^* \to \underline{s}} N_s^2(s^*)/N_s^1(s^*) = 0$

***Proof:*.** Let us start analyzing how the ratio $N_s^2(s^*)/N_s^1(s^*)$ depends on ϕ_0 and ϕ_1. The first thing to notice is that the second-order differential equation (18) satisfies the conditions for the continuity of the solution with respect to the boundary conditions.[22] Then $N_s^1(s^*)$ depends continuously on ϕ_0 and ϕ_1 while $N_s^2(s^*)$ depends continuously on ϕ_1; this in turn implies that the ratio depends continuously on ϕ_0 and ϕ_1. Now consider two possible initial conditions $\phi_0 > \phi_0'$. Then by Lemma 8.(i) we have $N_s^1(s^*) < N_s^{1\prime}(s^*)$ which in turn implies that $N_s^2(s^*)/N_s^1(s^*) > N_s^{2\prime}(s^*)/N_s^{1\prime}(s^*)$. Then $N_s^2(s^*)/N_s^1(s^*)$ is increasing in ϕ_0.

Now consider two possible middle conditions $\phi_1 > \phi_1'$. Then by Lemma 8.(i) we have $N_s^2(s^*) < N_s^{2\prime}(s^*)$ and by Lemma 8.(ii) we have $N_s^1(s^*) > N_s^{1\prime}(s^*)$, which in turn implies that the ratio $N_s^2(s^*)/N_s^1(s^*)$ is decreasing in ϕ_1.

To show that $N_s^2(s^*)/N_s^1(s^*)$ depends continuously on s^*, notice that the inverse of the matching function H satisfies

$$H(\phi) = H^1(\phi)\, 1_{[\phi_0,\phi_1)} + H^2(\phi)\, 1_{[\phi_1,\phi_2]}$$

where H^i satisfies the following second-order differential equation

$$H_{\phi\phi}(\phi) = \left[(\sigma - 1)\frac{A_\phi(H(\phi),\phi)}{A(H(\phi),\phi)} + \frac{g_\phi(\phi)}{g(\phi)}\right] H_\phi(\phi) - \left[\frac{A_s(H(\phi),\phi)}{A(H(\phi),\phi)} + \frac{V_s(H(\phi))}{V(H(\phi))}\right] H_\phi(\phi)^2$$

with boundary conditions $H^1(\phi_0) = \underline{s}$, $H^1(\phi_1) = H^2(\phi_1) = s^*$ and $H^2(\phi_2) = \overline{s}$. The previous differential equation also satisfies the conditions for the continuity of the solution with respect to the boundary conditions. Then the ratio $H_\phi^1(\phi_1)/H_\phi^2(\phi_1)$ depends continuously on s^*.

Now consider two possible values $s^* > s^{*\prime}$. The by Lemma 8.(i) we have $H_\phi^2(\phi_1) < H_\phi^{2\prime}(\phi_1)$ and by Lemma 8.(ii) $H_\phi^1(\phi_1) > H_\phi^{1\prime}(\phi_1)$. These two results yield $H_\phi^1(\phi_1)/H_\phi^2(\phi_1) > H_\phi^{1\prime}(\phi_1)/H_\phi^{2\prime}(\phi_1)$ which in turn implies that the ratio $H_\phi^1(\phi_1)/H_\phi^2(\phi_1)$ is increasing in s.

Finally notice that H^i is the inverse function of N^i, that is, $(N^i)^{-1} = H^i$ and so $H_\phi^1(\phi_1)/H_\phi^2(\phi_1) = N_s^2(s^*)/N_s^1(s^*)$ and so $N_s^2(s^*)/N_s^1(s^*)$ depends continuously on s^* and it is increasing in s^*.

Now I will show that $\lim_{\phi^+ \to \overline{\phi}} N_s^2(s^*)/N_s^1(s^*) = 0$. First, notice that $N_s^1(s^*) \to M_s^1(s^*) \in \mathbb{R}_{++}$ as $\phi_1 \to \phi_2$, where $M(s)$ is the solution to (18) with boundary conditions $M(\underline{s}) = \phi_0$ and $M(s^*) = \phi_2$. Then it only remains to show $\lim_{\phi_1 \to \phi_2} N_s^2(s^*) = 0$. So take any increasing sequence (ϕ_{0n}) such that $\phi_{0n} \to \phi_2$. We will have a corresponding sequence of solutions $(N^{2,n})$. By continuity on the boundary conditions, $N^{2,n} \to N^2$ and $N_s^{2,n} \to N_s^2$. Besides it is clear that $N^{2,n}(s) \to N^2(s) = \phi_2$ for all $s \in [\underline{s}, \overline{s}]$ and so $N_s^2(s) = 0$ for all $s \in [\underline{s}, \overline{s}]$.

[22] See Argarwal-Regan (2008) Chapter 12.

In a similar way it can be shown that $\lim_{\phi_1 \to \phi_0} N_s^2(s^*) \in \mathbb{R}_{++}$ and $\lim_{\phi_1 \to \phi_0} N_s^1(s^*) = 0$. In the case of $\lim_{s^* \to \overline{s}} N_s^2(s^*)/N_s^1(s^*) = \infty$ and $\lim_{s^* \to \underline{s}} N_s^2(s^*)/N_s^1(s^*) = 0$ we can repeat the analysis presented above for the inverse function H^i. ■

The purpose of the following Lemma is to analyze the effects of the introduction of a concave kink in the matching function at s^*. Let N be defined as in (35) with boundary conditions $N^1(\underline{s}) = \phi_0$, $N^1(s^*) = \phi_1^N = N^2(s^*)$ and $N^2(\overline{s}) = \phi_2$, and let M be a function that satisfies (18) with boundary conditions $M(\underline{s}) = \phi_0$ and $M(\overline{s}) = \phi_2$.[23]

Lemma 10 *Suppose that* $N_s^2(s^*)/N_s^1(s^*) = \alpha < 1$. *Then* $N(s) > M(s)$ *for all* $s \in (\underline{s}, \overline{s})$.

Proof:. Notice that we can write

$$M(s) = M^1(s)\, 1_{[\underline{s}, s^*)} + M^2(s)\, 1_{[s^*, \overline{s}]},$$

where the M^i's satisfy (18) with boundary conditions $M^1(\underline{s}) = \phi_0$, $M^1(s^*) = \phi_1^M = M^2(s^*)$ and $M^2(\overline{s}) = \phi_2$. Notice that since M is twice differentiable on $[\underline{s}, \overline{s}]$, ϕ_1^M must be such that $M_s^2(s^*)/M_s^1(s^*) = 1$. From Lemma 9.ii we know that the ratio of the right to left derivative at the s^* is decreasing in ϕ_1 and since

$$N_s^2(s^*)/N_s^1(s^*) = \alpha < 1 = M_s^2(s^*)/M_s^1(s^*)$$

we must have $\phi_1^M < \phi_1^N$. Then, a direct application of Lemma 8.i on M^2, N^2 and of Lemma 8.ii on M^1, N^1 yields the result, i.e, $N(s) > M(s)$ for all $s \in (\underline{s}, \overline{s})$. ■

Suppose that we have two functions N and M that satisfy

$$\begin{aligned} N(s) &= N^1(s)\, 1_{[\underline{s}, s_N^*)} + N^2(s)\, 1_{[s_N^*, \overline{s}]}, \\ M(s) &= M^1(s)\, 1_{[\underline{s}, s_M^*)} + M^2(s)\, 1_{[s_M^*, \overline{s}]}, \end{aligned}$$

where M^i, N^i satisfy (18) with boundary conditions $N^1(\underline{s}) = \phi_0^N$, $N^1(s_N^*) = \phi_1^N = N^2(s^*)$, $N^2(\overline{s}) = \phi_2$ and $M^1(\underline{s}) = \phi_0^M$, $M^1(s_M^*) = \phi_1^M = M^2(s_M^*)$, $M^2(\overline{s}) = \phi_2$. As discussed before, the functions N and M present kinks at s_N^* and s_M^*, respectively. The following Lemma compares the right–to-left derivative ratios at the kinks when the functions N and M cross as in Figure A.1.

[23] In this way M will be twice differentiable on $[\underline{s}, \overline{s}]$.

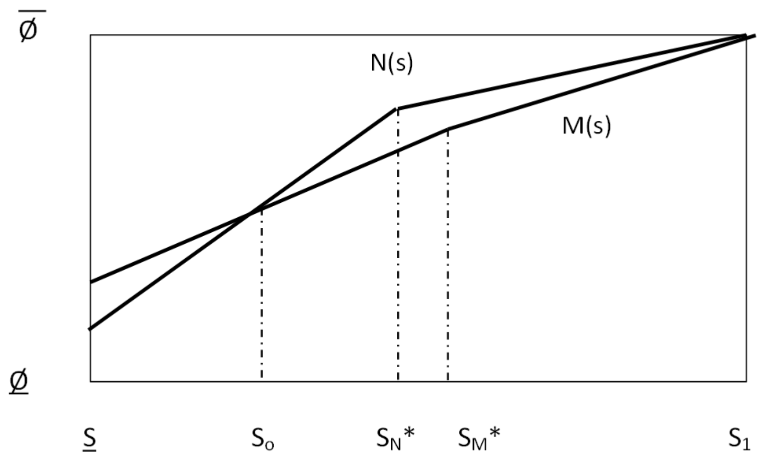


Figure A.1

Lemma 11 *Let N and M be defined as before. Let* $s_0, s_1 \in [\underline{s}, \overline{s}]$ *with* $s_0 < s_1$ *and suppose that the following conditions hold:*
(i) $N(s_0) = M(s_0)$ *and* $N(s_1) = M(s_1)$
(ii) $N(s) > M(s)$ *for all* $s \in (s_0, s_1)$
(iii) $s_N^*, s_M^* \in (s_0, s_1)$
Then $N_s^2(s_N^*)/N_s^1(s_N^*) < M_s^2(s_M^*)/M_s^1(s_M^*)$.

Proof:. Using (18) it can be shown that when condition (i) and (iii) holds then

$$\begin{aligned} \frac{N_s^2(s_N^*)}{N_s^1(s_N^*)} &= \frac{N_s(s_1)}{N_s(s_0)} \exp\left\{\sigma \int_{N(s_0)}^{N(s_1)} \frac{\partial \ln A(H^N(t), t)}{\partial \phi} dt\right\} B \\ \frac{M_s^2(s_M^*)}{M_s^1(s_M^*)} &= \frac{M_s(s_1)}{M_s(s_0)} \exp\left\{\sigma \int_{M(s_0)}^{M(s_1)} \frac{\partial \ln A(H^M(t), t)}{\partial \phi} dt\right\} B \end{aligned} \tag{36}$$

where H^i represents the inverse function of $i = N, M$ and

$$B = \frac{A(s_0, N(s_0))}{A(s_1, N(s_1))} \frac{V(s_0)}{V(s_1)} \frac{g(N(s_1))}{g(N(s_0))}$$

Conditions (i),(ii) and Lemma 8 imply

$$N_s(s_1)/N_s(s_0) < M_s(s_1)/M_s(s_0) \tag{37}$$

Because of condition (i), the limits of integration in the previous expressions are the same while condition (ii) implies that $H^N(t) < H^M(t)$ for all $t \in [N(s_0), N(s_1)]$. Combining this with the strict log-supermodularity of A we get

$$\exp\left\{\sigma\int_{N(s_0)}^{N(s_1)}\frac{\partial\ln A\left(H^N\left(t\right),t\right)}{\partial\phi}dt\right\}<\exp\left\{\sigma\int_{M(s_0)}^{M(s_1)}\frac{\partial\ln A\left(H^M\left(t\right),t\right)}{\partial\phi}dt\right\} \tag{38}$$

Finally, combining (36)-(38) we get

$$\frac{N_s^2\left(s_N^*\right)}{N_s^1\left(s_N^*\right)}<\frac{M_s^2\left(s_M^*\right)}{M_s^1\left(s_M^*\right)}$$

which is the desired result. ■

9 Appendix B

Proof of Lemma 2. First I will set some notation that will be used in the proof. Let $S(\phi) \equiv \{s \in S : l(s,\phi) > 0\}$ and let $\Phi(s) = \left\{\phi \in \left[\phi^*, \overline{\phi}\right] : l(s,\phi) > 0\right\}$. To clarify the exposition of the proof I will proceed in a series of steps.

STEP 1: $\Phi(s) \neq \emptyset$ for all $s \in S$ and $S(\phi) \neq \emptyset$ for all $\phi \in \left[\phi^*, \overline{\phi}\right]$.

The full employment condition (10) and $V(s) > 0$ directly imply $\Phi(s) \neq \emptyset$ for all $s \in S$. Now suppose that we have an equilibrium in which there is $\phi \in \left[\phi^*, \overline{\phi}\right]$ such that $S(\phi) = \emptyset$. Then from (3) we have $q(\phi) = 0$ and this in incompatible with the demand given in (1), since for any $p(\phi) \in \mathbb{R}_+$ we have $q(\phi) > 0$. Then in any equilibrium we must have $S(\phi) \neq \emptyset$.

STEP 2: $S(.)$ and $\Phi(.)$ satisfy the following properties: (i) if $s \in S(\phi)$, $s' \in S(\phi')$ and $\phi' > \phi$, then $s' \geq s$; and (ii) if $\phi \in \Phi(s)$, $\phi' \in \Phi(s')$ and $s' > s$,then $\phi' \geq \phi$.

(i) Suppose that this is not true and so let $s' < s$. Notice that (6) implies that $s \in S(\phi)$ and only if $s \in \arg\min_z w(z)/A(z,\phi)$. Then $w(s)/A(s,\phi) \leq w(s')/A(s',\phi)$. In a similar way, $s' \in S(\phi')$ implies $w(s')/A(s',\phi') \leq w(s)/A(s,\phi')$. Combining both inequalities we get $A(s,\phi')A(s',\phi) \leq A(s,\phi)A(s',\phi')$, but this contradicts the log-supermodularity of A (remember that $\phi' > \phi$ and $s > s'$). Then we must have $s' \geq s$.

(ii) Suppose that this is not true and so let $\phi' < \phi$. Then $\phi \in \Phi(s) \Rightarrow s \in S(\phi)$ and $\phi' \in \Phi(s') \Rightarrow s' \in S(\phi')$. Then we have $\phi' < \phi$, $s \in S(\phi)$, $s' \in S(\phi')$ and by STEP 2.i this implies $s \geq s'$, which is a contradiction. Then we must have $\phi' \geq \phi$.

STEP 3: (i) $S(\phi)$ is an interval for all $\left[\phi^*, \overline{\phi}\right]$ and $|S(\phi) \cap S(\phi')| \leq 1$ for any two different $\phi, \phi' \in \left[\phi^*, \overline{\phi}\right]$; (ii) $\Phi(s)$ is an interval for all $s \in S$ and $|\Phi(s) \cap \Phi(s')| \leq 1$ for any two different $s, s' \in S$.

(i) I will prove the first part by contradiction. Suppose there is $\phi \in \left[\phi^*, \overline{\phi}\right]$ such that $S(\phi)$ is not an interval. Then there we can find $s, s' \in S(\phi)$, with $s < s'$, and some $s'' \in (s, s')$ such that $s'' \notin S(\phi)$. From STEP 1 we know that $\Phi(s'')$ is nonempty and so there must be a $\phi'' \in \left[\phi^*, \overline{\phi}\right]$ such that $s'' \in S(\phi'')$. We have only two possibilities: $\phi'' > \phi$ and $\phi'' < \phi$. If $\phi'' > \phi$, then STEP 2.i implies $s'' \geq s'$ which is a contradiction. If $\phi'' < \phi$, then STEP 2.i implies $s \geq s''$ which is also a contradiction. Then $S(\phi)$ is an interval for all $\left[\phi^*, \overline{\phi}\right]$.

Let us now show that $S(\phi)$ is at most a singleton and as before I will proceed by contradiction. Suppose that the claim is not true. Then there must be $\phi, \phi' \in \left[\phi^*, \overline{\phi}\right]$ such that $s, s' \in S(\phi) \cap S(\phi')$ with $s \neq s'$. Without loss of generality assume $\phi' > \phi$ and $s' > s$. Then we have $\phi' > \phi$, $s' \in S(\phi)$, $s \in S(\phi')$ and so STEP 2.i implies $s \geq s'$ which is a contradiction. This concludes part i.

(ii) I prove this by contradiction. Suppose there is $s \in S$ such that $\Phi(s)$ is not an interval. Then there we can find $\phi, \phi' \in \Phi(s)$, with $\phi < \phi'$, and some $\phi'' \in (\phi, \phi')$ such that $\phi'' \notin \Phi(s)$. From STEP 1 we know that $S(\phi'')$ is nonempty and so there must be a $s'' \in S$ such that $\phi'' \in \Phi(s'')$. We have only two possibilities: $s'' > s$ and $s'' < s$. If $s'' > s$, then STEP 2.ii implies $\phi'' \geq \phi'$ which is a contradiction. If $s'' < s$, then STEP 2.ii implies $\phi \geq \phi''$ which is also a contradiction. Then $\Phi(s)$ is an interval for all $s \in S$.

Let us now show that $\Phi(\phi)$ is at most a singleton and as before I will proceed by contradiction. Suppose that the claim is not true. Then there must be $s, s' \in S$ such that $\phi, \phi' \in \Phi(s) \cap \Phi(s')$ with $\phi \neq \phi'$. Without loss of generality assume $\phi' > \phi$ and $s' > s$. Then we have $s' > s$, $\phi' \in \Phi(s)$, $\phi \in \Phi(s')$ and so STEP 2.ii implies $\phi \geq \phi'$ which is a contradiction. This concludes part ii.

STEP 4: $S(\phi)$ is a singleton for all but a countable subset of $[\phi^*, \overline{\phi}]$.

I show this by contradiction. Let $\Phi_0 = \{\phi \in [\phi^*, \overline{\phi}] : |S(\phi)| > 1\}$ and suppose Φ_0 is uncountable. Notice that STEP 3.i implies that $S(\phi)$ is a nondegenerate interval for all $\phi \in \Phi_0$. Then for each $\phi \in \Phi_0$ we can pick a rational skill $r(\phi) \in intS(\phi)$ and given that $|S(\phi) \cap S(\phi')| \leq 1$ for any two different ϕ, ϕ' we must have $r(\phi) \neq r(\phi')$ when $\phi \neq \phi'$. Then the function $r : \Phi_0 \to \mathbb{Q} \cap S$ defined before is injective and so it is a contradiction since Φ_0 is uncountable.

STEP 5: $\Phi(s)$ is a singleton for all but a countable subset of S.

This follows from the same arguments as in STEP 4.

STEP 6: $S(\phi)$ is a singleton for all $\phi \in [\phi^*, \overline{\phi}]$.

I proceed by contradiction. Suppose there is $\phi \in [\phi^*, \overline{\phi}]$ such that $S(\phi)$ is not a singleton. Then STEP 3.i implies that $S(\phi)$ is an interval. By STEP 5 $\Phi(s) = \{\phi\}$ for all but a countable subset of $S(\phi)$. Then

$$l(s, \phi) = V(s)\,\delta\left[1 - I_{S(\phi)}\right] \text{ for almost all } s \in S(\phi)$$

where δ is the Dirac delta function. But then $q(\phi) = \int_{s \in S(\phi)} A(s, \phi)\, l(s, \phi)\, ds = \infty$, and this is incompatible with an equilibrium (as defined above). In other words, if $S(\phi)$ is not a singleton, then we would have a positive mass of workers producing in a single type of productivity firms which are of mass zero, and this cannot happen in equilibrium.

STEP 7: $\Phi(s)$ is a singleton for all $s \in S$.

I proceed by contradiction. Suppose there is an $s \in S$ such that $\Phi(s)$ is not a singleton. Then STEP 3.ii implies that $\Phi(s)$ is an interval. By STEP 6 $S(\phi) = \{s\}$ for all $\phi \in \Phi(s)$. Now let $\Phi_0 \subseteq \Phi(s)$ be the set of productivity levels that are assign a strictly positive *conditional*[24] mass of s-skill workers. I will show that Φ_0 is at most countable. The total conditional mass of s-skill workers allocated to productivities in Φ_0 can be expressed as

$$\int_{\Phi_0} l(s, \phi)\, d\phi = \int_{\phi^*}^{\overline{\phi}} k(\phi)\, \delta[1 - I_{\Phi_0}] d\phi$$

where δ is the Dirac delta function and $k(\phi)$ is the conditional mass of worker at productivity $\phi \in \Phi_0$. Notice that $\Phi_0 = \cup_{n=1}^{\infty} \{\phi \in \Phi_0 : k(\phi) \geq 1/n\}$ and because of the full employment condition each

[24] Remeber that the mass of workers of a particular skill s is zero. However, conditional on the skill, we can think of $l(s, \phi)$ as the density that represents the distribution of workers with skill s among the firms indexed by the productivity level. Then conditional on skill s, all s-skill workers have a total mass $V(s) > 0$. Then I say that a set $A \subseteq [\phi^*, \overline{\phi}]$ has possitive *conditional* mass if

$$\int_{\phi \in A} l(s, \phi)\, d\phi > 0.$$

$\{\phi \in \Phi_0 : k(\phi) \geq 1/n\}$ must be finite. Then Φ_0 is at most countable. This means a zero conditional mass of s-skill workers are allocated to almost all $\phi \in \Phi(s)$, which in turn means that $q(\phi) = 0$ for almost all $\phi \in \Phi(s)$. However this is incompatible with equilibrium since for any $p(\phi) \in \mathbb{R}_+$, the demand of variety ϕ (according to (1)) is strictly positive.

Steps 1,6,7 imply that there is a bijection $N : S \to [\phi^*, \overline{\phi}]$ such that $l(s, \phi) > 0$ if and only if $\phi = N(s)$ and by STEP 2 it must be strictly increasing. ■

Proof of Lemma 3. Lets us start with the differentiability of the wage schedule. Notice that (6) and Lemma 2 implies $s = \arg\min_z w(z)/A(z, N(s))$. Then

$$\begin{aligned}
\frac{w(s)}{A(s, N(s))} &\leq \frac{w(s+ds)}{A(s+ds, N(s))} \\
\frac{w(s+ds)}{A(s+ds, N(s+ds))} &\leq \frac{w(s)}{A(s, N(s+ds))}
\end{aligned}$$

and combining both inequalities we get

$$\frac{A(s+ds, N(s))}{A(s, N(s))} \leq \frac{w(s+ds)}{w(s)} \leq \frac{A(s+ds, N(s+ds))}{A(s, N(s+ds))}$$

Finally, taking logs, dividing by ds and taking limits as $ds \to 0$ in the previous chain of inequalities[25], yields (11).

To show that the price function is differentiable I proceed in a similar way. Notice that (6) and Lemma 2 implies $\phi = \arg\max_\gamma p(\gamma) A(H(\phi), \gamma)$. Then

$$\begin{aligned}
p(\phi) A(H(\phi), \phi) &\geq p(\phi + d\phi) A(H(\phi), \phi + d\phi) \\
p(\phi + d\phi) A(H(\phi + d\phi), \phi + d\phi) &\geq p(\phi) A(H(\phi + d\phi), \phi)
\end{aligned}$$

and combining both inequalities we get

$$\frac{A(H(\phi), \phi + d\phi)}{A(H(\phi), \phi)} \leq \frac{p(\phi)}{p(\phi + d\phi)} \leq \frac{A(H(\phi + d\phi), \phi + d\phi)}{A(H(\phi + d\phi), \phi)}$$

Finally, taking logs,dividing by ds and taking limits as $ds \to 0$ in the previous chain of inequalities yields (12).

Let us now turn to condition (13). In equilibrium the demand of any variety ϕ should equal its supply:

$$RP^{\sigma-1} p(\phi)^{-\sigma} = \int_{s \in S} A(s, \phi) \, l(s, \phi) \, ds$$

and together with Lemma 2 the previous equation implies

$$l(s, \phi) = \frac{RP^{\sigma-1} p(\phi)^{-\sigma}}{A(s, \phi)} \delta[s - H(\phi)] \tag{39}$$

[25]Remember that all the functions involved are continuous.

where δ is the Dirac delta function and $H = N^{-1}$. Besides, the total mass of workers used in the production of varieties on the range $[\phi^*, \phi]$ should be equal to the total mass of workers used in those varieties implied by the matching function, that is, for all ϕ we must have

$$\begin{aligned} \int_{\phi^*}^{\phi} \left(\int_{s \in S} l\left(s, \phi'\right) ds \right) g\left(\phi'\right) d\phi' \overline{M} &= \int_{\underline{s}}^{H(\phi)} \left[L - fM\right] V\left(s\right) ds \\ \int_{\phi^*}^{\phi} \frac{RP^{\sigma-1} p\left(\phi'\right)^{-\sigma}}{A\left(H\left(\phi'\right), \phi'\right)} g\left(\phi'\right) d\phi' \overline{M} &= \int_{\underline{s}}^{H(\phi)} \left[L - fM\right] V\left(s\right) ds \end{aligned} \tag{40}$$

Notice that the left hand side (LHS) of (40) is the integral of a continuous function and then, when we consider the LHS as a function of the limit of integration ϕ, LHS(ϕ) is differentiable. Then the right hand side (RHS) must also be differentiable with respect to ϕ and this together with the continuity of V implies that $H(\phi)$ is differentiable. If we differentiate both sides with respect to ϕ we get that for all $\phi \in \left(\phi^*, \overline{\phi}\right)$

$$\frac{RP^{\sigma-1} p\left(\phi\right)^{-\sigma}}{A\left(H\left(\phi\right), \phi\right)} g\left(\phi\right) \overline{M} = \left[L - fM\right] V\left(H\left(\phi\right)\right) H_{\phi}\left(\phi\right) \tag{41}$$

Now that we know that $H(\phi)$ is differentiable, it is easy to see that the LHS of (41) is differentiable with respect to ϕ and so the RHS must also be differentiable and so $H_{\phi}(\phi)$ is differentiable. Given that $H = N^{-1}$ it is straight forward to see that $N(s)$ and $N_s(s)$ will also be differentiable. That is, N and H are twice differentiable. Changing variables in (41) and after some rearrangement gives (13).

Finally, condition (14) guarantees that firms with the cutoff productivity level ϕ^* make zero profits. ■

Matching function and Lorenz dominance. Consider two economies A and B with matching functions N^A, N^B and suppose that $N^B(s) > N^A(s)$ for all $s \in [s_0, s_1]$. Then from (15) and the strict log-supermodularity we have that for all $s' > s$ in $[s_0, s_1]$

$$\frac{w^A\left(s'\right)}{w^A\left(s\right)} < \frac{w^B\left(s'\right)}{w^B\left(s\right)}$$

In this context, the poorest ρ fraction of workers is associated with a skill $s(\rho)$ given by

$$\rho = \int_{s_0}^{s(\rho)} V\left(s\right) ds \Bigg/ \int_{s_0}^{s_1} V\left(s\right) ds$$

The Lorenz Curve is then

$$\begin{aligned} L(\rho) &= \int_{s_0}^{s(\rho)} w(s)\,V(s)\,ds \Big/ \int_{s_0}^{s_1} w(s)\,V(s)\,ds \\ &= \frac{\int_{s_0}^{s(\rho)} \frac{w(s)}{w(s(\rho))} V(s)\,ds}{\int_{s_0}^{s(\rho)} \frac{w(s)}{w(s(\rho))} V(s)\,ds + \int_{s(\rho)}^{s_1} \frac{w(s)}{w(s(\rho))} V(s)\,ds} \end{aligned}$$

It is readily seen that this implies that for all $\rho \in (0,1)$

$$L^A(\rho) > L^B(\rho)$$

Finally, from Atkinson (1970) we know that Lorenz dominance is equivalent to *Normalized* Second-Order Stochastic Dominance. ■

Proof of Lemma 4. Let us start with the "only if" part of the lemma. Taking logs on both sides of (13), differentiating with respect to s and using (12) yields (18). This last equation is an ordinary second-order differential equation (SODE) that the matching function must satisfy with boundary conditions $N(\underline{s}) = \phi^*$, $N(\overline{s}) = \overline{\phi}$. Equation (19) guarantees that the value of the wages paid by firms to the workers used in the variable production is equal to the wages that these workers get when the firms with productivity ϕ^* are making zero profits. To see this remember that with a CES demand structure, variable costs are a constant fraction of total revenue

$$w(H(\phi))\,l(\phi) = \frac{\sigma-1}{\sigma} r(\phi),$$

where $l(\phi)$ is the total number of variable-production workers hired by firm with productivity ϕ. Then the total value of wages paid by firms is given by

$$\begin{aligned} \int_{\phi^*}^{\overline{\phi}} w(H(\phi))\,l(\phi)\,g(\phi)\,d\phi\overline{M} &= \frac{\sigma-1}{\sigma}\int_{\phi^*}^{\overline{\phi}} r(\phi)\,g(\phi)\,d\phi\overline{M} \\ &= \frac{(\sigma-1)}{\sigma} r(\phi^*) \int_{\phi^*}^{\overline{\phi}} e^{(\sigma-1)\int_{\phi^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} g(\phi)\,d\phi\overline{M} \end{aligned} \tag{42}$$

where the second line is obtained using (17). On the other hand, the total value of variable-production workers' wages is

$$\int_{\underline{s}}^{\overline{s}} [L - fM]\,w(s)\,V(s)\,ds = \left[L - [1 - G(\phi^*)]\,f\overline{M}\right] \tag{43}$$

In equilibrium we must have

$$\int_{\phi^*}^{\overline{\phi}} w(H(\phi))\,l(\phi)\,g(\phi)\,d\phi\overline{M} = \int_{\underline{s}}^{\overline{s}} [L - fM]\,w(s)\,V(s)\,ds$$

and using (42), (43) we get

$$\overline{M}\frac{(\sigma-1)}{\sigma}r(\phi^*)\int_{\phi^*}^{\overline{\phi}}\left[e^{(\sigma-1)\int_{\phi^*}^{\phi}\frac{\partial\ln A(H(t),t)}{\partial\phi}dt}+\frac{\sigma}{\sigma-1}\frac{f}{r(\phi^*)}\right]g(\phi)\,d\phi=L \tag{44}$$

Finally, using $r(\phi^*)=\sigma f$ (condition (14)) on the previous expression we get (19).

Let us turn to the "if" part of the Lemma. I will show that for a given ϕ^*, if a function $N:S\to[\phi^*,\overline{\phi}]$ satisfies (18), then we can construct quantities, prices and wages such that conditions (11)-(13) are satisfied. Suppose then that $N:S\to[\phi^*,\overline{\phi}]$ satisfies the conditions stated in the Lemma. The first thing to notice is that once we know $N(s)$ we can recover the quantity produced by each firm:

$$q(N(s))=\frac{A(s,N(s))\,[L-fM]\,V(s)}{g(N(s))\,\overline{M}N_s(s)} \tag{45}$$

and this in turn determines the consumption aggregator Q.

Now I will show that when the matching function satisfies (18) and the price schedule satisfy (12), then condition (13) will be satisfied irrespective of the level of the price schedule. Notice that if prices satisfy (12) then the left and right hand sides of (13) have the same instantaneous growth rate when the matching function satisfies (18) and so, for any two skill levels s,s' we will have

$$\frac{q(N(s'))}{q(N(s))}=\left(\frac{p(N(s'))}{p(N(s))}\right)^{-\sigma} \tag{46}$$

where $q(N(s))$ is given by (45). Then it is enough to show that (13) is satisfied at $\underline{s}$. Taking $q(\phi^*)$ out of the integral in the definition of Q and using (46) we get

$$Q=q(\phi^*)\,P_0^{-\sigma}$$

where P_0 is the aggregate price level when $p(\phi^*)=1$; using this in (13) we get

$$\begin{aligned}q(N(\underline{s})) &= QP^{\sigma}p(N(s))^{-\sigma}\\ &= q(N(\underline{s}))\,P_0^{-\sigma}P_0^{\sigma}p(N(s))^{\sigma}\,p(N(s))^{-\sigma}\\ &= q(N(\underline{s}))\end{aligned}$$

and so the condition is satisfied.

Finally, it is readily seen that if condition (19) holds, then from (44) it must be the case that $r(\phi^*)=\sigma f$ and so condition (14) is also satisfied.

The question of existence and uniqueness of the autarky equilibrium reduces to the question of existence and uniqueness of a solution (ϕ^*,N) to the system of equations (18)-(19). For any given ϕ^*, the conditions for existence and uniqueness of the solution to the boundary value problem including the second order ordinary differential equation (18) are satisfied. Then, for each exit cutoff ϕ^* there will be a unique matching function $N(s;\phi^*)$ satisfying (18). Notice that a direct application of Lemma 8.i implies that if $\phi^{*\prime\prime} > \phi^{*\prime}$ then $H\left(\phi;\phi^{*\prime\prime}\right) < H\left(\phi;\phi^{*\prime}\right)$ for all $\phi \in [\phi^{*\prime\prime},\overline{\phi})$.[26] This fact and the strict log-supermodularity of A implies that the left hand side of (19) is strictly decreasing on ϕ^* and it is equal to zero when $\phi^* = \overline{\phi}$. Then if

$$\overline{M}\left(\sigma-1\right) f \int_{\underline{\phi}}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\underline{\phi}}^{\phi} \frac{\partial \ln A\left(H\left(t;\underline{\phi}\right),t\right)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g\left(\phi\right) d\phi > L$$

there exist a unique equilibrium. I will elaborate a little bit more on the intuition behind the previous condition. Consider an equilibrium where the cutoff ϕ^* is given, that is, an equilibrium where the active firms in the market is given. In that equilibrium the profits of the firms with productivity ϕ^* are not necessarily equal to zero and so ϕ^* has to adjust to the point where $\pi\left(\phi^*\right) = 0$. It can be shown that as ϕ^* decreases, $\pi\left(\phi^*\right)$ decreases. If the mass of potential firms is not sufficiently high relative to the mass of workers, it is possible that even when all potential firms are active ($\phi^* = \underline{\phi}$), the profits of the exit cutoff firms are still positive. Because of this, there may not exit an equilibrium in which $\pi\left(\phi^*\right) = 0$. ∎

Before continuing, I will define some notation to clarify the exposition. Define the functions

$$R\left(\phi^*,\phi_x^*,s_x^*\right) \equiv \frac{N_s^x\left(s_x^*\right)}{N_s^d\left(s_x^*\right)} \tag{47}$$

$$E\left(\phi^*,\phi_x^*,s_x^*\right) \equiv e^{(\sigma-1)\int_{\phi^*}^{\phi_x^*} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} \tag{48}$$

$$\begin{aligned} W\left(\phi^*,\phi_x^*,s_x^*\right) \equiv{} & \overline{M}\left(\sigma-1\right) f \int_{\phi^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g\left(\phi\right) d\phi + \\ & \cdots + \overline{M}\left(\sigma-1\right) f_x \int_{\phi_x^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi_x^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g\left(\phi\right) d\phi \end{aligned} \tag{49}$$

Then equations (32)-(34) can be written as

$$\begin{aligned} R\left(\phi^*,\phi_x^*,s_x^*\right) &= \frac{1}{1+\tau^{1-\sigma}} \\ E\left(\phi^*,\phi_x^*,s_x^*\right) &= \tau^{\sigma-1} f_x / f \\ W\left(\phi^*,\phi_x^*,s_x^*\right) &= L \end{aligned}$$

This notation makes explicit the dependence of the left hand side of equations (32)-(34) on the cutoffs ϕ^*,ϕ_x^*,s_x^*. In Lemma (9) we provided a complete analysis of the function R and now we will study the behavior of the functions E with respect to ϕ^* and s_x^*. The following Lemma summarizes the most relevant features for our purposes.

[26] Lemma 8.i directly implies $N\left(s;\phi^{*\prime\prime}\right) > N\left(s;\phi^{*\prime}\right)$ for all $s \in [\underline{s},\overline{s})$ wich in turn implies the above condition on the inverse functions.

Lemma 12 *Let E be the function defined above. Then E is continuous, strictly increasing in s_x^* and strictly decreasing in ϕ^*.*

Proof. The first thing to notice is that the second-order differential equation (18) satisfies the conditions for the continuity of the solution with respect to the boundary conditions and so the matching function N and its inverse H depend continuously on the parameters. Then from (33) it is clear that E is continuous.

Let N,N' be the matching functions that are obtained from the triplets $(\phi^*, \phi_x^*, s_x^*)$, $(\phi^*, \phi_x^*, s_x^{*\prime})$ respectively, where $s_x^* < s_x^{*\prime}$.[27] Applying Lemma 8.ii to the inverse matching functions H and H' we conclude that $H'(t) > H(t)$ for all $t \in (\phi^*, \phi_x^*]$. This fact and the strict log-supermodularity of A imply $\int_{\phi^*}^{\phi_x^*} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt > \int_{\phi^*}^{\phi_x^*} \frac{\partial \ln A(H'(t),t)}{\partial \phi} dt$ which in turn imply $E(\phi^*, \phi_x^*, s_x^*) < E(\phi^*, \phi_x^*, s_x^{*\prime})$.

Let N,N' be the matching functions that are obtained from the triplets $(\phi^*, \phi_x^*, s_x^*)$, $\left(\phi^{*\prime}, \phi_x^*, s_x^*\right)$ respectively, where $\phi^* < \phi^{*\prime}$. Lemma 8 implies that the situation is the one depicted in Figure B.1. As we can see from the picture, $H(t) > H'(t)$ for all $t \in [\phi^{*\prime}, \phi_x^*]$. Similarly to the previous case, this last fact and the strict log-supermodularity of A imply $E(\phi^*, \phi_x^*, s_x^*) > E\left(\phi^{*\prime}, \phi_x^*, s_x^*\right)$. ■

Proof of Lemma 7. Let us turn here to the issue of existence and uniqueness of the equilibrium. The goal here is analyze under what condition the system (18),(32)-(34) has a unique solution, or in other words, under what conditions we have a unique equilibrium featuring selection into trade. As discussed in the text, a necessary condition to have an equilibrium featuring selection into trade is $\tau^{\sigma-1} f_x/f > 1$. However, this condition is not sufficient: given that the support of the productivity distribution is bounded above, then nobody exports if $\tau^{\sigma-1} f_x/f$ is high enough. Since an equilibrium featuring selection into trade does not exists for all combinations of τ and f_x, it is helpful to analyze what kind of equilibrium emerges under different combinations of τ and f_x (with $f_x > 0$) and study existence and uniqueness in each case.

Let us define

$$a \equiv E\left(\phi_a^*, \overline{\phi}, \overline{s}\right) = e^{(\sigma-1)\int_{\phi_a^*}^{\overline{\phi}} \frac{\partial \ln A\left(H^a(t),t\right)}{\partial \phi} dt} > 1 \tag{50}$$

where ϕ_a^*, H^a are the exit productivity cutoff and the inverse of the matching function in the autarky equilibrium.

Claim 7.1: *If $\tau^{\sigma-1} f_x/f \leq 1$, then there exists a unique equilibrium in which all producers export.*

Suppose that $\tau^{\sigma-1} f_x/f \leq 1$. In this case as in the closed economy, we only have one relevant cutoff, i.e., the exit productivity cutoff ϕ^*. Using the same arguments as in the closed economy case, it can be shown that in equilibrium, the matching function and the exit cutoffs satisfy (18) and

$$\overline{M}(\sigma-1)(f+f_x)\int_{\phi^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma-1}\right] g(\phi)\, d\phi = L. \tag{51}$$

Notice that the only difference with the system (18),(19) is that now the fix cost is augmented by

[27] Recall that the cutoffs are sufficient statistics for the matching function.

f_x in equation (51). From the proof of Lemma 4 we know that the left hand side of (51) is strictly decreasing in ϕ^* and it is equal to zero when $\phi^* = \overline{\phi}$. Then, if

$$\overline{M}\left(\sigma-1\right)\left(f+f_x\right)\int_{\underline{\phi}}^{\overline{\phi}}\left[e^{(\sigma-1)\int_{\underline{\phi}}^{\phi}\frac{\partial \ln A\left(H\left(t;\underline{\phi}\right),t\right)}{\partial\phi}dt}+\frac{1}{\sigma-1}\right]g\left(\phi\right)d\phi > L,$$

there exist a unique equilibrium. Clearly, the exit cutoff in the autarky equilibrium is strictly less than in the trade equilibrium in which everybody exports. This concludes the proof of Claim 1.

Claim 7.2: *If $\tau^{\sigma-1}f_x/f \geq a$, then there is a unique equilibrium in which nobody exports.*

The first thing to notice is that this combination of parameters does not allow for an equilibrium in which everybody exports. To see this notice that $\tau^{\sigma-1}f_x/f \geq a > 1$ implies that it is not profitable for exit-cutoff firms to export.

Now I will show that there is no equilibrium in which some producers export. In particular, I will show that there is no solution to the system (18),(32)-(34) for this combination of parameters. I will proceed by contradiction. Suppose that there is a trade equilibrium and let $N^t, \phi^*, \phi_x^*, s_x^*$ be a solution to (18),(32)-(34). If we let ϕ_a^*, N^a be the exit productivity cutoff and matching function in the autarky equilibrium, then Lemma 8 implies that there are only four possible configurations of N^t and N^a as it is shown in Figure B.4. Panels (a) and (b) can be ruled out using the same arguments as in STEP 1 of the proof of Theorem 1.

Let us consider panel (c) now. To simplify the exposition let us define the function F^t, F^a as follows

$$\begin{aligned} F^t\left(\phi\right) &= \int_{\phi^*}^{\phi}\frac{\partial \ln A\left(H^t\left(t\right),t\right)}{\partial\phi}dt \\ F^a\left(\phi\right) &= \int_{\phi_a^*}^{\phi}\frac{\partial \ln A\left(H^a\left(t\right),t\right)}{\partial\phi}dt \end{aligned}$$

The functions F^t, F^a are continuous and strictly increasing in ϕ and satisfy

$$F^t\left(\phi_x^*\right) = E\left(\phi^*, s_x^*, \phi_x^*\right) = \tau^{\sigma-1}f_x/f \geq a = E\left(\phi_a^*, \overline{s}, \overline{\phi}\right) \geq F^a\left(\overline{\phi}\right)$$

In what follows I will show that $F^t\left(\phi\right) > F^a\left(\phi\right)$ for all $\phi \geq \phi_a^*$. Since F^t, F^a are strictly increasing, $F^t\left(\phi_x^*\right) = F^a\left(\overline{\phi}\right)$, then

$$F^t\left(\overline{\phi}\right) > F^a\left(\overline{\phi}\right) \tag{52}$$

Now let ϕ_1 be the productivity level at which the matching function intersect and let $\phi \in [\phi_a^*, \phi_1]$. In this case we have $H^t\left(t\right) > H^a\left(t\right)$ for all $t \in [\phi_a^*, \phi)$. Then strict log-supermodularity of A and $\phi_a^* > \phi^*$ imply $F^t\left(\phi\right) > F^a\left(\phi\right)$. Then we conclude that $F^t\left(\phi\right) > F^a\left(\phi\right)$ for all $\phi \in [\phi_a^*, \phi_1]$.

Finally let $\phi \in \left(\phi_1, \overline{\phi}\right)$ and suppose $F^t(\phi) \leq F^a(\phi)$. Notice that

$$\begin{aligned} F^t\left(\overline{\phi}\right) &= F^t(\phi) + \int_{\phi}^{\overline{\phi}} \frac{\partial \ln A\left(H^t(t), t\right)}{\partial \phi} dt \\ F^a\left(\overline{\phi}\right) &= F^a(\phi) + \int_{\phi}^{\overline{\phi}} \frac{\partial \ln A\left(H^a(t), t\right)}{\partial \phi} dt \end{aligned}$$

In this case we have $H^t(t) < H^a(t)$ for all $t \in [\phi, \overline{\phi})$. Then strict log-supermodularity of A immediately implies

$$\int_{\phi}^{\overline{\phi}} \frac{\partial \ln A\left(H^a(t), t\right)}{\partial \phi} dt > \int_{\phi}^{\overline{\phi}} \frac{\partial \ln A\left(H^t(t), t\right)}{\partial \phi} dt$$

which in turn implies $F^a\left(\overline{\phi}\right) > F^t\left(\overline{\phi}\right)$. But this contradicts (52). Then $F^t(\phi) > F^a(\phi)$ for all $\phi \in \left(\phi_1, \overline{\phi}\right)$.

In the preceding paragraphs we have shown that $F^t(\phi) > F^a(\phi)$ for all $\phi \in [\phi_a^*, \overline{\phi}]$ and this in turn implies that the first term in the left hand side of (34) is strictly greater than the left hand side of (19) and given that the second term in the left hand side of the former equation is nonnegative we get that (34),(19) cannot hold simultaneously. But this contradicts that N^t and N^a are equilibrium matching functions ruling out panel (c).

Let us now turn to panel (d). Notice that in this case $H^t(t) < H^a(t)$ for all $t \in [\phi^*, \phi_x^*]$. Then the strict log-supermodularity of A together with the definition of a and equation (33) imply

$$\tau^{\sigma-1} \frac{f_x}{f} = e^{(\sigma-1) \int_{\phi^*}^{\phi_x^*} \frac{\partial \ln A\left(H^t(t), t\right)}{\partial \phi} dt} < e^{(\sigma-1) \int_{\phi_a^*}^{\overline{\phi}} \frac{\partial \ln A\left(H^a(t), t\right)}{\partial \phi} dt} = a$$

which is a contradiction.

Given that every possible configuration lead us to a contradiction, we conclude that there is no solution to the system (18),(32)-(34) when $\tau^{\sigma-1} f_x / f \geq a$. In this case only an autarky equilibrium is possible, and by Lemma 4 we know that the equilibrium exists and it is unique. This concludes the proof of Claim 2.

Claim 7.3: *If* $1 < \tau^{\sigma-1} f_x / f < a$, *then there is a unique trade equilibrium that features selection into trade.*

The proof of this claim is long and tedious. For that reason I will give a brief outline of the proof first and then I will go into the details. The first thing to notice is that if $(\phi^*, \phi_x^*, s_x^*)$ is a solution to (32)-(34), then (ϕ_x^*, s_x^*) is a solution to (32),(33) when the exit cutoff is ϕ^*. Then I first analyze this smaller system of equations for each value of ϕ^*. To that end I construct the functions g, h that summarize the link between ϕ_x^* and s_x^* contained in equations (32), (33) respectively and then I study the conditions for existence and uniqueness of the solution to (32),(33) given ϕ^*, represented by certain types of intersections of g and h.

The result of the analysis above is a pair of functions $\phi_x^*(\phi^*)$ and $s_x^*(\phi^*)$, that represent the solution of the system (32),(33) given ϕ^*. I will use these functions in equation (34) and show that the equation has a unique solution ϕ^*. Finally, the triple $(\phi^*, \phi_x^*(\phi^*), s_x^*(\phi^*))$ is the unique solution to the system (32)-(34).

Let us now get into the details. To clarify the exposition I will proceed in a series of steps.

STEP 1: Analysis of equation (32) and definition of the function g.

Let us fix ϕ^*. Then Lemma 9 implies that for each $\phi_x^* \in (\phi^*, \overline{\phi})$ [28], there is a unique $s_x^*(\phi_x^*)$ that solves equation (32) and that the function $s_x^*(\phi_x^*)$ is strictly increasing. Notice that for a given s_x^*, $\lim_{\phi_x^* \to \overline{\phi}} R(\phi^*, \phi_x^*, s_x^*) = 0$ and $\lim_{\phi_x^* \to \phi^*} R(\phi^*, \phi_x^*, s_x^*) = \infty$ and this implies that $\lim_{\phi_x^* \to \overline{\phi}} s_x^*(\phi_x^*) = \overline{s}$ and $\lim_{\phi_x^* \to \phi^*} s_x^*(\phi_x^*) = \underline{s}$. Now define the function $g: [\phi^*, \overline{\phi}] \to [\underline{s}, \overline{s}]$ given by

$$g(\phi_x^*; \phi^*) = \begin{cases} \underline{s} & \text{if } \phi_x^* = \phi^* \\ s_x^*(\phi_x^*) & \text{if } \phi_x^* \in (\phi^*, \overline{\phi}) \\ \overline{s} & \text{if } \phi_x^* = \overline{\phi} \end{cases}$$

This concludes step 1.

STEP 2: Analysis of equation (33) and definition of the function h.

Since E is strictly increasing in s_x^* [29], for each ϕ_x^*, there is at most one solution $s_x^*(\phi_x^*)$ to equation (33). However, it may be the case that for a given ϕ_x^*, there is no solution to (33) in $[\underline{s}, \overline{s}]$. Then, let $\Phi_0 \subseteq [\phi^*, \overline{\phi}]$ be the set of values of ϕ_x^* for which a solution s_x^* to (33) exist in $[\underline{s}, \overline{s}]$ and define the continuous function

$$h(\phi_x^*; \phi^*) = \begin{cases} s_x^*(\phi_x^*) & \text{if } \phi_x^* \in \Phi_0 \\ \overline{s} & \text{if } \phi_x^* \in [\phi^*, \overline{\phi}] \setminus \Phi_0 \end{cases}$$

Notice that $E(\phi^*, \phi^*, s_x^*) = 1 < \tau^{\sigma-1} f_x / f$ for all $s_x^* \in [\underline{s}, \overline{s}]$ and so $h(\phi^*; \phi^*) = \overline{s}$. Besides, by construction we have $h(\overline{\phi}; \phi^*) \leq \overline{s}$. This concludes step 2.

STEP 3: The functions g and h defined above intersect exactly once on $[\phi^*, \overline{\phi}]$. [30]

Let us start with the existence of the intersection. From steps 1 and 2 we have $h(\phi^*; \phi^*) = \overline{s} > \underline{s} = g(\phi^*; \phi^*)$, $h(\overline{\phi}; \phi^*) \leq \underline{s} = g(\overline{\phi}; \phi^*)$ and together with the continuity of g and h imply that there is some $\phi_x^* \in [\phi^*, \overline{\phi}]$ such that $g(\phi_x^*; \phi^*) = h(\phi_x^*; \phi^*)$.

Let us now turn to the uniqueness of the intersection. First, notice that in any intersection, the intersection pair (ϕ_x^*, s_x^*) must belong to the graph of g where $s_x^* = g(\phi_x^*; \phi^*) = h(\phi_x^*; \phi^*)$. This fact together with $h(\phi^*; \phi^*) = \overline{s} > \underline{s} = g(\phi^*; \phi^*)$ imply that the intersection pair (ϕ_x^*, s_x^*) must satisfy: $(\phi_x^*, s_x^*) \in (\phi^*, \overline{\phi}) \times (\underline{s}, \overline{s})$ or $(\phi_x^*, s_x^*) = (\overline{\phi}, \overline{s})$. Besides, by construction of the functions g and h, equations (32),(33) must hold for (ϕ_x^*, s_x^*) when $(\phi_x^*, s_x^*) \in (\phi^*, \overline{\phi}) \times (\underline{s}, \overline{s})$.

[28] Notice that, $R(\phi^*, \overline{\phi}, s_x^*) = 0$ for all $s_x^* \in (\underline{s}, \overline{s})$ and it is not defined for $s \in \{\underline{s}, \overline{s}\}$, and so there is no solution s_x^* to (32). For the case of $\phi_x^* = \phi^*$, Notice that $R(\phi^*, \phi^*, s_x^*)$ is not defined for $s \in [\underline{s}, \overline{s}]$ and so there is not solution to (32) either.

[29] See Lemma above.

[30] See figure B.2 for the two possible kind of intersections of g and h.

Now suppose that the functions have more than one intersection and let $\left(\phi_x^{*\prime}, s_x^{*\prime}\right) \neq \left(\phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right)$ be two of its intersections. Let $\left(\phi^*, \phi_x^{*\prime}, s_x^{*\prime}\right)$, $\left(\phi^*, \phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right)$ be the corresponding cutoffs. Without loss of generality, the discussion in the previous paragraph leaves only two possibilities: (i) $\left(\phi_x^{*\prime}, s_x^{*\prime}\right)$, $\left(\phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right) \in \left(\phi^*, \overline{\phi}\right) \times (\underline{s}, \overline{s})$ or (ii) $\left(\phi_x^{*\prime}, s_x^{*\prime}\right) \in \left(\phi^*, \overline{\phi}\right) \times (\underline{s}, \overline{s})$ and $\left(\phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right) = \left(\overline{\phi}, \overline{s}\right)$. In what follows I will show that in each case we arrive to a contradiction.

(i) Let N' and N'' be the matching functions corresponding to $\left(\phi^*, \phi_x^{*\prime}, s_x^{*\prime}\right)$ and $\left(\phi^*, \phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right)$ respectively. Lemma 8 implies that there are only two possibilities concerning the position of N' and N'': (a) one curve is above the other for all $s \in (\underline{s}, \overline{s})$ as depicted in panel (a) of Figure B.3; or (b) the curves cross as in panel (b) of Figure B.3.

(a) Without loss of generality suppose that $N'(s) > N''(s)$ for all $s \in (\underline{s}, \overline{s})$ as in the figure. Notice that all the conditions in Lemma 11 are satisfied and so this implies $R\left(\phi^*, \phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right) > R\left(\phi^*, \phi_x^{*\prime}, s_x^{*\prime}\right)$. But this means that given ϕ^*, equation (32) cannot hold for $\left(\phi_x^{*\prime}, s_x^{*\prime}\right)$ and $\left(\phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right)$ simultaneously which is a contradiction.

(b) Now suppose that the N' and N'' cross as in panel b of Figure B.3. Notice that $H''(t) < H'(t)$ for all $t \in \left[\phi^*, \phi_x^{*\prime\prime}\right]$. This fact together with the strict log-supermodularity of A imply $E\left(\phi^*, \phi_x^{*\prime}, s_x^{*\prime}\right) > E\left(\phi^*, \phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right)$. But this implies that (33) cannot hold simultaneously for $\left(\phi_x^{*\prime}, s_x^{*\prime}\right)$ and $\left(\phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right)$ and this is a contradiction.

(ii) In this case we have $\left(\phi_x^{*\prime}, s_x^{*\prime}\right) \in \left(\phi^*, \overline{\phi}\right) \times (\underline{s}, \overline{s})$ and $\left(\phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right) = \left(\overline{\phi}, \overline{s}\right)$. Notice that all the conditions in Lemma 10 are satisfied and so we must have $N'(s) > N''(s)$ for all $s \in (\underline{s}, \overline{s})$. The situation is depicted in panel (c) of Figure B.3. First, by construction of the function h we have $E\left(\phi^*, \phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right) = E\left(\phi^*, \overline{\phi}, \overline{s}\right) \leq \tau^{\sigma-1} f_x/f$. Second, notice that $H'(t) < H''(t)$ for all $t \in \left[\phi^*, \phi_x^{*\prime}\right]$. This, together with the strict log-supermodularity of A imply $E\left(\phi^*, \phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right) > E\left(\phi^*, \phi_x^{*\prime}, s_x^{*\prime}\right)$. Third, given that $\left(\phi_x^{*\prime}, s_x^{*\prime}\right) \in \left(\phi^*, \overline{\phi}\right) \times (\underline{s}, \overline{s})$ we have that equation (33) hold for the pair $\left(\phi_x^{*\prime}, s_x^{*\prime}\right)$ which means that $E\left(\phi^*, \phi_x^{*\prime}, s_x^{*\prime}\right) = \tau^{\sigma-1} f_x/f$. Finally, putting all the inequalities together yields

$$\tau^{\sigma-1} f_x/f = E\left(\phi^*, \phi_x^{*\prime}, s_x^{*\prime}\right) < E\left(\phi^*, \phi_x^{*\prime\prime}, s_x^{*\prime\prime}\right) \leq \tau^{\sigma-1} f_x/f$$

and this is a contradiction.

Given that in every possible case we arrive to a contradiction, we conclude that there cannot be more that one intersection. The two kinds of intersections are depicted in Figure B.2. This concludes step 3.

For any given ϕ^*, if (ϕ_x^*, s_x^*) is a solution to (32),(33), then (ϕ_x^*, s_x^*) must be an intersection of the functions g and h constructed above. However, the converse is not true, i.e., if (ϕ_x^*, s_x^*) is an intersection of g and h then (ϕ_x^*, s_x^*) is not necessarily a solution to (32),(33). In particular, when the intersection satisfies $(\phi_x^*, s_x^*) \in \left(\phi^*, \overline{\phi}\right) \times (\underline{s}, \overline{s})$, then by construction of the functions g, h, (ϕ_x^*, s_x^*) must also be a solution to (32),(33). However, when $(\phi_x^*, s_x^*) = \left(\overline{\phi}, \overline{s}\right)$, the system (32),(33) has no solution. The previous discussion implies that the system (32),(33) has at most one solution and that the existence of a solution depends on the value of ϕ^*. The next step analyzes the range of values of ϕ^* for which the solution exist.

STEP 4: There is a cutoff value $b > \phi_a^*$[31] such that (32),(33) has no solution for $\phi^* \geq b$ and has a unique solution for $\phi^* < b$.

Let us define b as the value that satisfies

$$E\left(b, \overline{\phi}, \overline{s}\right) \equiv e^{(\sigma-1)\int_b^{\overline{\phi}} \frac{\partial \ln A\left(H^b(t),t\right)}{\partial \phi} dt} = \tau^{\sigma-1} f_x / f \tag{53}$$

where H^b is the inverse of the matching function N^b that satisfies (18) on $[\underline{s}, \overline{s}]$ with boundary conditions $N^b(\underline{s}) = b$, $N^b(\overline{s}) = \overline{\phi}$. By Lemma 12 E is strictly decreasing in ϕ^*. Then (50) and (53) imply $b > \phi_a^*$, since by assumption $\tau^{\sigma-1} f_x/f < a$[32].

From discussion preceding this step, it is enough to show that the unique intersection of g and h, (ϕ_x^*, s_x^*) satisfies $(\phi_x^*, s_x^*) = \left(\overline{\phi}, \overline{s}\right)$ for $\phi^* \geq b$ and $(\phi_x^*, s_x^*) \in \left(\phi^*, \overline{\phi}\right) \times (\underline{s}, \overline{s})$ for $\phi^* < b$. So let $\phi^* \geq b$. Given that E is strictly decreasing in ϕ^* and strictly increasing in s_x^*, (53) implies $E\left(\phi^*, \overline{\phi}, s_x^*\right) \leq \tau^{\sigma-1} f_x/f$ for all $s \in [\underline{s}, \overline{s}]$. Then by definition of h we have $h\left(\overline{\phi}; \phi^*\right) = \overline{s}$ for all $\phi^* \geq b$. This fact and $g\left(\overline{\phi}; \phi^*\right) = \overline{s}$ imply $(\phi_x^*, s_x^*) = \left(\overline{\phi}, \overline{s}\right)$.

Now let $\phi^* < b$. Notice that to prove that the intersection satisfies $(\phi_x^*, s_x^*) \in \left(\phi^*, \overline{\phi}\right) \times (\underline{s}, \overline{s})$ it is enough to show that $h\left(\overline{\phi}; \phi^*\right) < \overline{s}$. Notice that (53) and the fact that E is strictly decreasing in ϕ^* imply $E\left(\phi^*, \overline{\phi}, \overline{s}\right) > \tau^{\sigma-1} f_x/f$. Given that E is strictly increasing in s_x^* we have $E\left(\phi^*, \overline{\phi}, s_x^*\right) = \tau^{\sigma-1} f_x/f$ for some $s_x^* < \overline{s}$. Finally, by definition of h we have $h\left(\overline{\phi}; \phi^*\right) < \overline{s}$. This concludes step 4.

STEP 5: Let $(\phi_x^*(\phi^*), s_x^*(\phi^*))$ be the intersection for a given ϕ^*. Then the function $\overline{W}(\phi^*) \equiv W(\phi^*, \phi_x^*(\phi^*), s_x^*(\phi^*))$[33] is continuous on $\left[\underline{\phi}, \overline{\phi}\right]$ and strictly decreasing in ϕ^* on $[\underline{\phi}, b)$.[34]

It is clear that equations (32),(33) are continuous on ϕ^* and so the solution $(\phi_x^*(\phi^*), s_x^*(\phi^*))$ is also continuous on ϕ^*. Finally, given that $W(\phi^*, \phi_x^*, s_x^*)$ is continuous on the cutoffs we have that $\overline{W}(\phi^*)$ is continuous.

Let $\phi^{*\prime}, \phi^{*\prime\prime} \in [\underline{\phi}, b)$ with $\phi^{*\prime\prime} > \phi^{*\prime}$, let $\left(s_x^{*\prime}, \phi_x^{*\prime}\right), \left(s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ be the corresponding intersections and let N', N'' be the matching functions obtained from the cutoffs $\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right), \left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ respectively. Recall that $\phi^{*\prime}, \phi^{*\prime\prime} \in [\underline{\phi}, b)$ implies by Step 4 that equations (32),(33) hold, i.e.,

$$R\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right) = R\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right) = 1/\left(1 + \tau^{1-\sigma}\right) \tag{54}$$

$$E\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right) = E\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right) = \tau^{\sigma-1} f_x/f \tag{55}$$

The matching function N' and the exit cutoff $\phi^{*\prime\prime}$ are depicted in Figure B.X.a. The kink of N'' given by $\left(s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ has to be located in one of the regions A,B,C described in the picture. Suppose that $\left(s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ is in region A. Then all the conditions of Lemma 10 are satisfied and so $R\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right) < R\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$. But this contradicts (54).

Now suppose $\left(s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ is in region B. The situation is depicted in Figure B.X.b. As we can see,

[31] ϕ_a^* is the autarky exit productivity cutoff.
[32] See Claim 3.
[33] See (49).
[34] In fact $\overline{W}$ is strictly increasing on $\left[\underline{\phi}, \overline{\phi}\right]$, but the statement in this step is enough to our purposes.

$H''(t) < H'(t)$ for all $t \in [\phi^{*\prime\prime}, \phi_x^{*\prime\prime}]$. The strict log-supermodularity of A and $[\phi^{*\prime\prime}, \phi_x^{*\prime\prime}] \subset [\phi^{*\prime}, \phi_x^{*\prime}]$ imply $E\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right) > E\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$. But this contradicts (55).

Given that the kink cannot be in regions A or B, then it must be the case that the kink $\left(s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ is in region C. I will show that when this is the case, then $\overline{W}\left(\phi^{*\prime}\right) > \overline{W}\left(\phi^{*\prime\prime}\right)$. When $\left(s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ is in region C, Lemma 8 implies that we have only two possibilities: (i) N' and N' do not cross on $[\underline{s}, \overline{s})$ as depicted in Figure B.X.c; or (ii) N' and N' cross as depicted in Figure B.X.d.

The first thing to notice is in both that cases $H''(\phi) < H'(\phi)$ for all $\phi \geq \phi_x^{*\prime\prime}$ and $\phi_x^{*\prime\prime} \geq \phi_x^{*\prime}$. These facts together with the strict log-supemodularity of A imply

$$\int_{\phi_x^{*\prime}}^{\phi} \frac{\partial \ln A\left(H'(t), t\right)}{\partial \phi} dt > \int_{\phi_x^{*\prime\prime}}^{\phi} \frac{\partial \ln A\left(H''(t), t\right)}{\partial \phi} dt$$

for all $\phi \geq \phi_x^{*\prime\prime}$. This in turn implies that the second term in (49) for $W\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right)$ is strictly greater than the corresponding term for $W\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$. Now I will show that in each case, the first term of $W\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right)$ is also strictly greater that the corresponding term of $W\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$.

(i) In this case $H''(t) < H'(t)$ for all $\phi \geq \phi^{*\prime\prime}$ and $\phi^{*\prime\prime} > \phi^{*\prime}$. Then the strict log-supemodularity of A implies

$$\int_{\phi^{*\prime}}^{\phi} \frac{\partial \ln A\left(H'(t), t\right)}{\partial \phi} dt > \int_{\phi^{*\prime\prime}}^{\phi} \frac{\partial \ln A\left(H''(t), t\right)}{\partial \phi} dt$$

for all $\phi \geq \phi^{*\prime\prime}$. This in turn implies that the first term in (49) for $W\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right)$ is strictly greater than the corresponding term for $W\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$.

(ii) To simplify the exposition let us define the function F', F'' as follows

$$\begin{aligned} F'(\phi) &= \int_{\phi^{*\prime}}^{\phi} \frac{\partial \ln A\left(H'(t), t\right)}{\partial \phi} dt \\ F''(\phi) &= \int_{\phi^{*\prime\prime}}^{\phi} \frac{\partial \ln A\left(H''(t), t\right)}{\partial \phi} dt \end{aligned}$$

The functions F', F'' are continuous and strictly increasing in ϕ and satisfy $F'\left(\phi_x^{*\prime}\right) = E\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right)$, $F''\left(\phi_x^{*\prime\prime}\right) = E\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$. This and (55) imply $F'\left(\phi_x^{*\prime}\right) = F''\left(\phi_x^{*\prime\prime}\right)$. In what follows I will show that $F'(\phi) > F''(\phi)$ for all $\phi \geq \phi^{*\prime\prime}$. Since F', F'' are strictly increasing, $F'\left(\phi_x^{*\prime}\right) = F''\left(\phi_x^{*\prime\prime}\right)$ and $\phi_1 \in \left(\phi_x^{*\prime}, \phi_x^{*\prime\prime}\right)$ then

$$F'(\phi_1) > F''(\phi_1) \tag{56}$$

. Now let $\phi \in \left(\phi_1, \overline{\phi}\right)$ and notice

$$\begin{aligned} F'(\phi) &= F'(\phi_1) + \int_{\phi_1}^{\phi} \frac{\partial \ln A\left(H'(t), t\right)}{\partial \phi} dt \\ F''(\phi) &= F''(\phi_1) + \int_{\phi_1}^{\phi} \frac{\partial \ln A\left(H''(t), t\right)}{\partial \phi} dt \end{aligned}$$

Given that $H'(t) > H''(t)$ for all $t \in (\phi_1, \phi)$, the strict log-supermodularity of A implies

$$\int_{\phi_1}^{\phi} \frac{\partial \ln A(H'(t), t)}{\partial \phi} dt > \int_{\phi_1}^{\phi} \frac{\partial \ln A(H''(t), t)}{\partial \phi} dt$$

and so $F'(\phi) > F''(\phi)$. Then we conclude that $F'(\phi) > F''(\phi)$ for all $\phi \geq \phi_1$.

Now let $\phi \in \left[\phi^{*\prime\prime}, \phi_0\right]$. In this case we have $H'(t) > H''(t)$ for all $t \in [\phi^{*\prime\prime}, \phi_0)$. Then the strict log-supermodularity of A immediately implies $F'(\phi) > F''(\phi)$. Then $F'(\phi) > F''(\phi)$ for all $\phi \in \left[\phi^{*\prime\prime}, \phi_0\right]$.

Finally let $\phi \in (\phi_0, \phi_1)$ and suppose $F'(\phi) \leq F''(\phi)$. Notice that

$$\begin{aligned} F'(\phi_1) &= F'(\phi) + \int_{\phi}^{\phi_1} \frac{\partial \ln A(H'(t), t)}{\partial \phi} dt \\ F''(\phi_1) &= F''(\phi) + \int_{\phi}^{\phi_1} \frac{\partial \ln A(H''(t), t)}{\partial \phi} dt \end{aligned}$$

In this case we have $H'(t) < H''(t)$ for all $t \in [\phi, \phi_1)$. Then strict log-supermodularity of A immediately implies

$$\int_{\phi}^{\phi_1} \frac{\partial \ln A(H''(t), t)}{\partial \phi} dt > \int_{\phi}^{\phi_1} \frac{\partial \ln A(H'(t), t)}{\partial \phi} dt$$

which in turn implies $F''(\phi_1) > F'(\phi_1)$. But this contradicts (56). Then $F'(\phi) > F''(\phi)$ for all $\phi \in (\phi_0, \phi_1)$.

The previous paragraphs imply $F'(\phi) > F''(\phi)$ for all $\phi \in [\phi^{*\prime\prime}, \overline{\phi}]$. Using this in (49) we conclude that the first term of $W\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right)$ is strictly greater than the corresponding term of $W\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$.

The previous analysis suggest that in any case, both terms of $W\left(\phi^{*\prime}, s_x^{*\prime}, \phi_x^{*\prime}\right)$ are strictly greater that the corresponding terms of $W\left(\phi^{*\prime\prime}, s_x^{*\prime\prime}, \phi_x^{*\prime\prime}\right)$ and from this we conclude

$$\overline{W}\left(\phi^{*\prime}\right) > \overline{W}\left(\phi^{*\prime\prime}\right)$$

that is, $\overline{W}(\phi^*)$ is strictly decreasing in ϕ^*. This concludes step 5.

STEP 6: There is a unique ϕ^* such that $\overline{W}(\phi^*) = L$.

Notice that when $\phi^* = b$[35], step 4 implies $\phi_x^*(b) = \overline{\phi}$ and $s_x^*(b) = \overline{s}$. Evaluating $\overline{W}$ at b yields

$$\overline{W}(b) = \overline{M}(\sigma - 1) f \int_b^{\overline{\phi}} \left[e^{(\sigma-1)\int_b^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma - 1} \right] g(\phi) d\phi$$

[35] b is defined in (53).

Then $b > \phi_a^*$ and (19) imply

$$\overline{W}(b) < \overline{M}(\sigma-1) f \int_b^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi^*}^{\phi} \frac{\partial \ln A(H(t),t)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g(\phi)\, d\phi = L$$

Finally, if $\overline{W}\left(\underline{\phi}\right) > L$ then step 5 implies that there is a unique $\phi^* \in \left(\underline{\phi}, b\right)$ such that $\overline{W}(\phi^*) = L$. This concludes step 6.

To conclude that proof of Claim 3, let ϕ^* be the unique value of the exit productivity cutoff that satisfies $\overline{W}(\phi^*) = L$. Then $(\phi^*, \phi_x^*(\phi^*), s_x^*(\phi^*))$ is the unique solution to system (32)-(34) where $(\phi_x^*(\phi^*), s_x^*(\phi^*))$ is the intersection of the the functions g and h defined above given ϕ^*, i.e., $(\phi_x^*(\phi^*), s_x^*(\phi^*))$ is the solution to (32),(33) given ϕ^*. This concludes the proof of Claim 3. ■

Proof of Theorem 1:. Let N^a, ϕ_a^* be the matching function and exit cutoff in the autarky equilibrium and let N^t,ϕ^*,s_x^*,ϕ_x^* be the matching function, the exit productivity cutoff, the skill export cutoff and the productivity export cutoff in the trade equilibrium. Remember that the trade matching function satisfies

$$N^t = N^{td} 1_{[\underline{s}, s_x^*)} + N^{tx} 1_{[s_x^*, \overline{s}]}$$

To clarify the exposition of the proof I will proceed in two steps.

STEP 1: The functions N^a, N^t must satisfy

$$\phi_x^* = N^t(s_x^*) > N^a(s_x^*)$$

To show this I will proceed by contradiction. Suppose that this is not the case and we have $N^t(s_x^*) \le N^a(s_x^*)$. If $N^t(s_x^*) = N^a(s_x^*)$ then Lemma 8 implies that $N^t(s) = N^a(s)$ for all $s \in [s_x^*, \overline{s}]$ and this situation is depicted in Figure B.5.a. If $N^t(s_x^*) < N^a(s_x^*)$ then Lemma 8.i implies $N^{tx}(s) < N^a(s)$ for all $s \in [s_x^*, \overline{s})$ and Lemma 10 implies that the $N^{td}(s) < N^a(s)$ for all $s \in [\underline{s}, s_x^*)$. This situation is depicted in Figure B.5.b. Because of the strict log-supermodularity of A we have that in any of the cases

$$\int_{\phi^*}^{\phi} \frac{\partial \ln A\left(H^t(t), t\right)}{\partial \phi} dt > \int_{\phi_a^*}^{\phi} \frac{\partial \ln A\left(H^a(t), t\right)}{\partial \phi} dt$$

for all $\phi \in \left[\phi_a^*, \overline{\phi}\right]$. Combining this with the fact that $\phi_a^* > \phi^*$ we obtain

$$\int_{\phi^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi^*}^{\phi} \frac{\partial \ln A\left(H^t(t),t\right)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g(\phi)\, d\phi > \int_{\phi_a^*}^{\overline{\phi}} \left[e^{(\sigma-1)\int_{\phi_a^*}^{\phi} \frac{\partial \ln A\left(H^a(t),t\right)}{\partial \phi} dt} + \frac{1}{\sigma-1} \right] g(\phi)\, d\phi$$

that is, the first term in the left hand side of (34) is strictly greater than the left hand side of (19) and given that the second term in the left hand side of the former equation is nonnegative we get that (34),(19) cannot hold simultaneously. This contradict that N^t and N^a are equilibrium matching functions, and so it must be the case that $N^t(s_x^*) > N^a(s_x^*)$. This means that the relative position of the matching functions must be represented by Figure B.5.c or B.5.d. The following step rules out panel c.

STEP 2: The autarky and trade exit cutoff satisfy $\phi_a^* < \phi^*$.

To show this I will proceed by contradiction. Suppose that we have $\phi_a^* \geq \phi^*$. Then STEP 1 and Lemma 8 imply that N^a,N^t will cross exactly once on the interval $[\underline{s}, s_x^*]$. The typical situation is depicted in Figure B.5.c[36]. Let $s_+ \in [\underline{s}, s_x^*)$ be the skill level at which the matching functions intersect and let $\phi_+ = N^a(s_+) = N^t(s_+)$. I will show that when this is the case, two alternative ways to compute the revenues of a firm with productivity ϕ_+ will yield different results.

Using (17),(14) and (30) we can express the revenues of a firm with productivity ϕ_+ as follows

$$r^i(\phi_+) = \sigma f \exp\left\{(\sigma-1)\int_{\phi_i^*}^{\phi_+} \frac{\partial \ln A\left(H^i(t), t\right)}{\partial \phi} dt\right\}$$

for $i = a, t$. Because of the strict log-supermodularity of A we have $r^t(\phi^+) > r^a(\phi^+)$ if $\phi^* < \phi_a^*$ and $r^t(\phi^+) = \sigma f = r^a(\phi^+)$ if $\phi^* = \phi_a^*$. In any case we must have

$$r^t(\phi^+) \geq r^a(\phi^+) \tag{57}$$

Let us now consider an alternative way to compare the revenues of firms with productivity ϕ_+. First I will compare the output produced by firms with productivity ϕ_+ in the autarky and trade equilibrium. They are given by

$$\begin{aligned} q^t(\phi^*) &= \frac{A(s_+, \phi_+)\left[L - [1-G(\phi^*)] f\overline{M} - [1-G(\phi_x^*)] f_x \overline{M}\right] V(s_+)}{g(\phi_+)\overline{M} N_s^t(s_+)} \\ q^a(\phi^*) &= \frac{A(s_+, \phi_+)\left[L - [1-G(\phi_a^*)] f\overline{M}\right] V(s_+)}{g(\phi_+)\overline{M} N_s^a(s_+)} \end{aligned}$$

Notice that Lemma 8.ii implies that $N_s^t(s_+) > N_s^a(s_+)$. This fact together with $\phi^* \leq \phi_a^*$ and $[1-G(\phi_x^*)] f_x \overline{M} > 0$ imply

$$q^t(\phi^*) < q^a(\phi^*) \tag{58}$$

Let us now compare the prices set by firms with productivity ϕ_+ in the autarky and trade equilibrium. They are given by

$$\begin{aligned} p^t(\phi_+) &= \frac{\sigma}{\sigma-1}\frac{w^t(s_+)}{A(s_+, \phi_+)} \\ p^a(\phi_+) &= \frac{\sigma}{\sigma-1}\frac{w^a(s_+)}{A(s_+, \phi_+)} \end{aligned}$$

From the previous expressions for prices it is clear that in order to compare the autarky and trade

[36] The figure depicts the case where $\phi^* < \phi_a^*$.

price of a firm with productivity ϕ^{+} it is enough to compare the autarky and trade wage of a worker with skill s_{+}. Because we set the average wage as the numeraire we have

$$\overline{w}^{i} = \int_{\underline{s}}^{\overline{s}} w^{i}(s) V(s)\, ds = 1$$

and taking $w(s_{+})$ out of the integral we get

$$w^{i}(s_{+}) \left[\int_{\underline{s}}^{s_{+}} \frac{w^{i}(s)}{w^{i}(s_{+})} V(s)\, ds + \int_{s_{+}}^{\overline{s}} \frac{w^{i}(s)}{w^{i}(s_{+})} V(s)\, ds \right] = 1$$

As was explained in the paragraphs following equation (15), the situation depicted in Figure B.5.c implies that the high-to-low wage ratios increase in the interval $[s_{+}, \overline{s}]$ and decrease in the interval $[\underline{s}, s_{+}]$. Then we must have

$$\int_{\underline{s}}^{s_{+}} \frac{w^{t}(s)}{w^{t}(s_{+})} V(s)\, ds + \int_{s_{+}}^{\overline{s}} \frac{w^{t}(s)}{w^{t}(s_{+})} V(s)\, ds > \int_{\underline{s}}^{s_{+}} \frac{w^{a}(s)}{w^{a}(s_{+})} V(s)\, ds + \int_{s_{+}}^{\overline{s}} \frac{w^{a}(s)}{w^{a}(s_{+})} V(s)\, ds$$

which in turn implies $w^{t}(s_{+}) < w^{a}(s_{+})$ and

$$p^{t}(\phi_{+}) < p^{a}(\phi_{+}) \tag{59}$$

Finally, combining (58) and (59) we get

$$r^{t}(\phi^{*}) < r^{a}(\phi^{*}) \tag{60}$$

Conditions (57) and (60) are a contradiction and so it must be the case that $\phi_{a}^{*} < \phi^{*}$.

Steps 1 and 2 imply that the situation must be like the one depicted in Figure B.5.d, that is

$$N^{t}(s) > N^{a}(s) \text{ for all } s \in [\underline{s}, \overline{s})$$

QED. ■

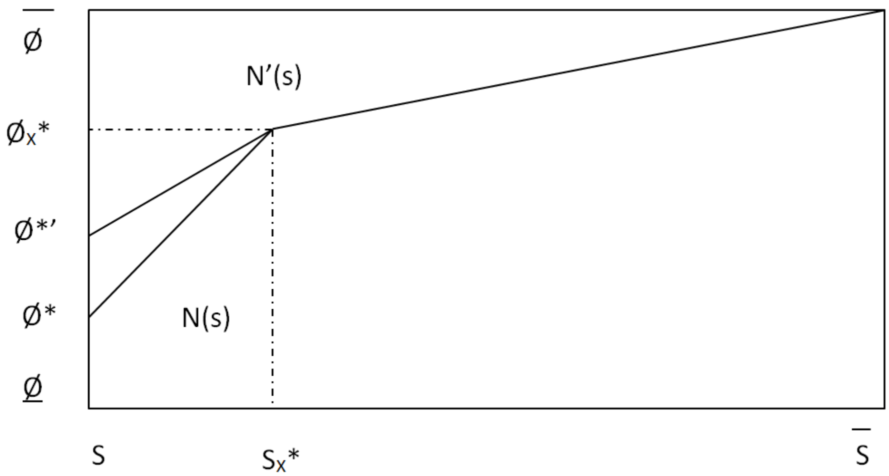
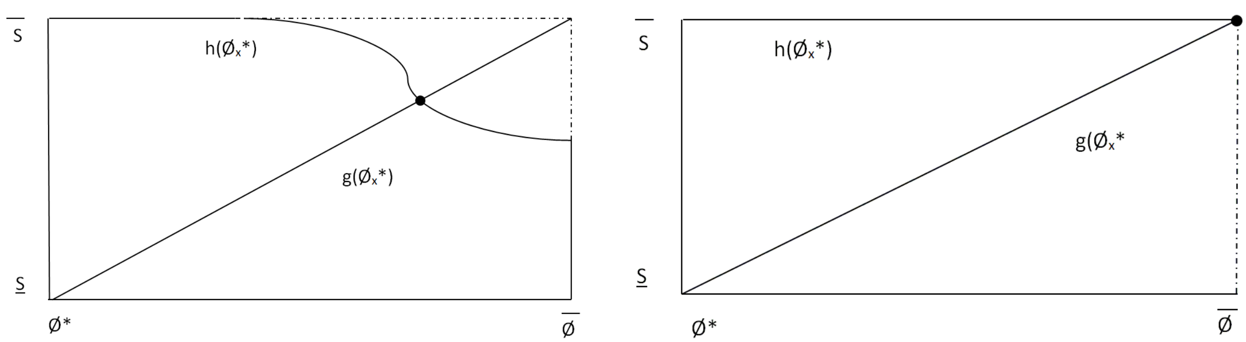
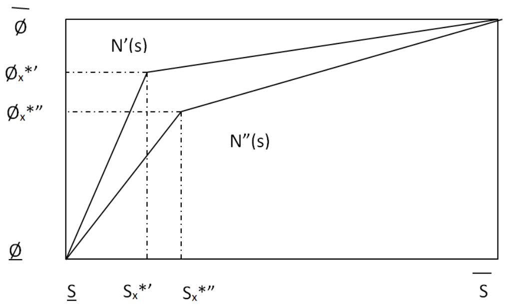


Figure B.1

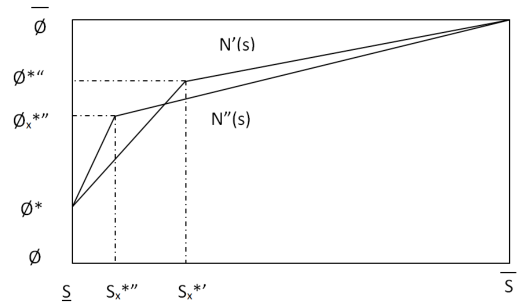


(a) (b)

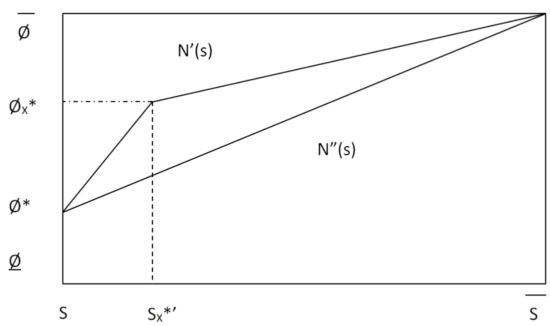
Figure B.2



(a)



(b)



(c)

Figure B.3

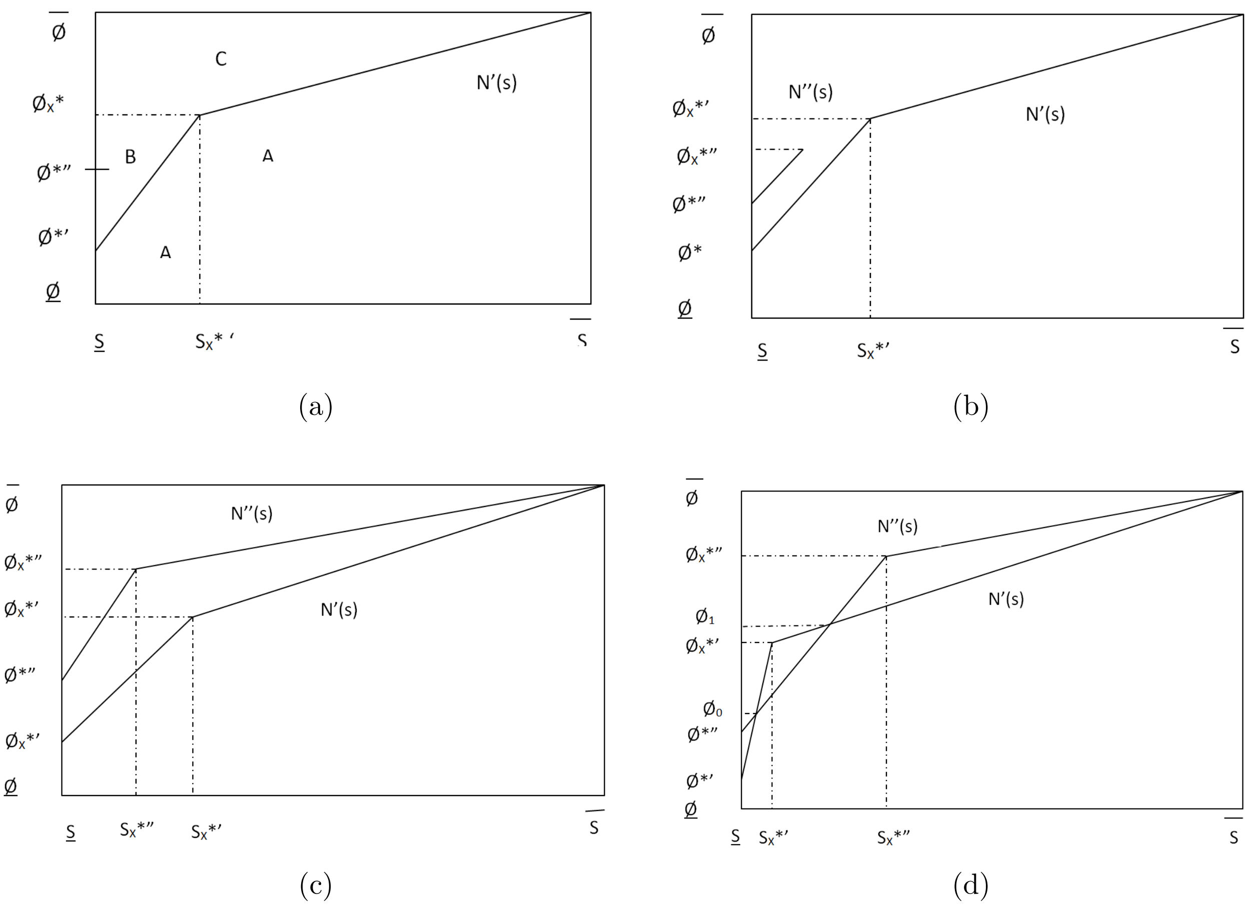
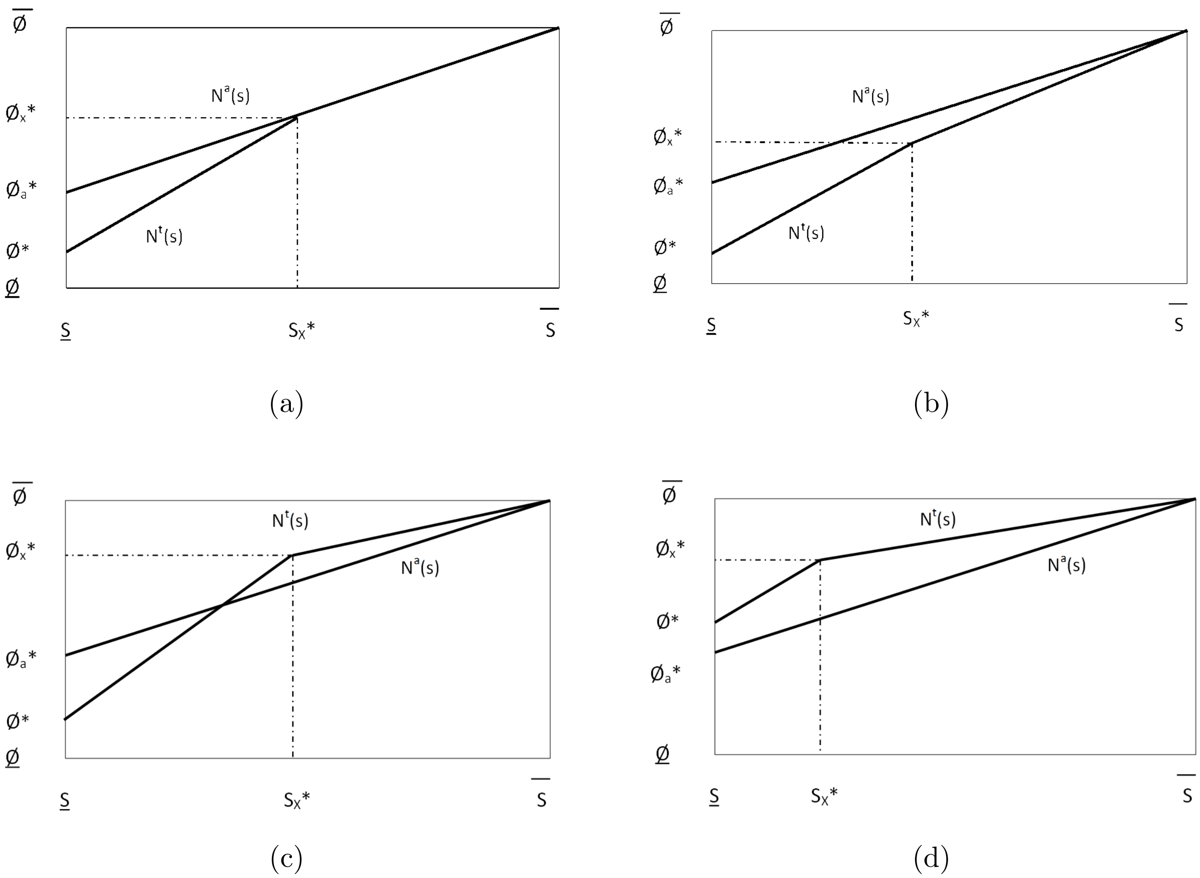


Figure B.4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(d)

Figure B.5

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